1. Let $S$ be a bounded set in $\mathbb{R}$. Then $Bd(S)$ is a closed set since it is the intersection of $Cl(S)$ and $Cl(\mathbb{R}\setminus S)$, both closed.

To see $Bd(S)$ is bounded, first we show $Cl(S)$ is bounded. There is a constant $r$ such that $|x| \leq r$ for all $x \in S$ (since $S$ bounded). If $y \in Cl(S)$, then there is an $x \in S$ with $|x - y| < 1$. Then $|y| \leq |y - x| + |x| < 1 + r$ for all $y \in Cl(S)$. Thus $Cl(S)$ is bounded. Now since $Bd(S) \subseteq Cl(S)$, $Bd(S)$ is also bounded. Thus $Bd(S)$ is compact since it is closed and bounded in $\mathbb{R}$.

2. Let $A, B \subseteq X$ where $A$ is closed and $B$ is compact. Then $A \cap B$ is closed since $B$ is closed. Now take any open cover for $A \cap B$, say $Q$. Then $Q \cup X \setminus (A \cap B)$ is an open cover for $B$ since $X \setminus (A \cap B)$ is open and $B \subseteq Q \cup (X \setminus (A \cap B))$. Then, since $B$ is compact, there is a finite subcover of $Q$ for $B$, say $Q'$. Then $Q' \setminus (X \setminus (A \cap B))$ is a finite subcover of $Q$ for $A \cap B$. Hence $A \cap B$ is compact.

3. First assume $X$ is compact and let $\{A_i\}_i$ be an arbitrary collection of closed sets with finite intersection property. We need to show that $\bigcap A_i \neq \emptyset$. Suppose $\bigcap A_i = \emptyset$. Then, $\bigcup (X \setminus A_i) = X$ where each $X \setminus A_i$ is open. So $\{X \setminus A_i\}_i$ is an open cover for $X$. Since $X$ is compact, there is a finite subcover of $\{X \setminus A_i\}_i$ for $X$, say $\{X \setminus A_i\}_{i=1,..,K}$. But then $X = \bigcup_{i=1,..,K} X \setminus A_i$. Thus, $\bigcap_{i=1,..,K} A_i = \emptyset$ which contradicts the finite intersection property of $\{A_i\}_i$.

Now suppose that every collection of closed set with finite intersection has a nonempty intersection. Suppose $X$ is not compact. Then there exists an open cover for $X$ which doesn’t have a finite subcover. Say, $\{B_i\}_i$ is an open cover for $X$, that is, $X = \bigcup B_i$. Then for any positive integer $K$, $\{B_i\}_{i=1,..,K}$ does not cover $X$, that is, $\bigcup_{i=1,..,K} B_i \neq X$. Therefore, $\bigcap_{i=1,..,K} X \setminus B_i \neq \emptyset$ for any $K$. Thus, $\{X \setminus B_i\}$ has finite intersection property. Since each $X \setminus B_i$ is closed, by assumption above we get $\bigcap \{X \setminus B_i\} \neq \emptyset$, that is, $X \neq \bigcup B_i$, which contradicts our supposition that $\{B_i\}_i$ is an open cover for $X$. Thus $X$ is compact.

4. Let $(X, d)$ be a complete metric space, and $S \subseteq X$.

(a) If $S$ is compact, then it is closed.

Since compactness implies precompactness, it is also precompact.

Let $S$ be closed and precompact. Since any closed subset of a complete metric space is also complete, $S$ is complete. But by the theorem, any complete and precompact set is compact, we get $S$ is compact.

(b) Note that $Cl(S)$ is closed. So if we show $Cl(S)$ is precompact, we are done by part (a) above. Let $\epsilon > 0$. Since $S$ is precompact, it can be covered by finitely many $\epsilon/2$-neighborhoods. That is, there exists a finite set $x_1, x_2, ..., x_n \subseteq S$ such that $S \subseteq \bigcup N_{\epsilon/2}(x_i)$. Note, if $x \in Cl(S)$, then there exists $y \in S$ with $d(x, y) < \epsilon/2$. 


There is at least one $x_i \in x_1, \ldots, x_n$ such that $y \in N_\epsilon(x_i)$. So by triangle inequality, we have $d(x, x_i) \leq d(x, y) + d(y, x_i) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus, $x \in N_\epsilon(x_i)$. Since this can be done for every $x \in Cl(S)$, we get $Cl(S) \subseteq \bigcup N_\epsilon(x)$. So $Cl(S)$ is precompact.

5. Let $(X, d)$ be a metric space in which every bounded sequence has a convergent subsequence.

To show that $X$ is complete, we need to show that every Cauchy sequence in $X$ converges to a point in $X$. Let $(x_n)$ be a Cauchy sequence in $X$. Then $(x_n)$ is bounded. By our supposition, it has a convergent subsequence, say $x_{k_n}$ converges to $x \in X$. Thus every Cauchy sequence in $X$ converges to a point in $X$, so $X$ is complete.

6. Let $(X, d)$ be a metric space where $d$ is the discrete metric. Then $X$ is complete. To see this, we’ll show that every Cauchy sequence converges in $X$. Let $(x_n)$ in $X$ be a Cauchy sequence. Then, for all $\epsilon > 0$ there exists $N_0 \ s.t \ d(x_k, x_l) < \epsilon$ for all $k, l \geq N_0$. Say $\epsilon = 1/2$. So $d(x_k, x_l) < 1/2$ for all $k, l \geq N_0$. In particular, $d(x_{N_0}, x_l) < 1/2$ for all $l \geq N_0$. Since $d$ is discrete, $d(x_{N_0}, x_l) = 0$, that is, $x_{N_0} = x_l$ for all $l \geq N_0$. That is, the sequence $(x_n)$ is equal to $x_{N_0}$ past the $N_0^{th}$ term. Thus $\lim x_n = x_{N_0} \in X$.

7. Fixed point of $\Gamma$: $x \in \Gamma(x)$.

Fixed set of Gamma: $S = \Gamma(S)$.

(a) Let $\Gamma : [1, 2] \Rightarrow [1, 2]$ such that $\Gamma(x) = [1, 2] \setminus \{x\}$ for all $x \in [1, 2]$. Then, for $S = [1, 2]$ we have $\Gamma(S) = [1, 2] = S$. So $S$ is a fixed set. But there is no fixed point.

(b) Let $x \in \Gamma(x)$. If $\Gamma(x)$ is singleton then we are done $S = x = \Gamma(S)$. If $\Gamma(x)$ is not singleton then for each $y \in \Gamma(x)$ with $y \neq x$ consider $\Gamma(y)$ and take the union over such $y$’s. Thus look at $\bigcup \Gamma(y)$. Now consider those $x' \notin \Gamma(x)$. Take the union $\bigcup \Gamma(x')$. Repeat this until $S = \Gamma(S)$ where $S$ is the set of all such $y$’s and $x'$’s.

8. If $\Gamma$ has a closed-graph then, since $Y$ is compact, $\Gamma$ will be upper-hemicontinuous by proposition. So, all we need to show is that $\Gamma$ has a closed-graph, that is, $\Gamma$ is closed at any $x \in X$. To see this, suppose that $x_n \to x$ in $X$, $y_n \to y$ in $Y$ with $y_n \in \Gamma(x_n)$ for all $n$.

We need to show $y \in \Gamma(x)$, that is, $d_y(y, f(x)) \leq \epsilon$. Note that $d(y_n, f(x_n)) \leq \epsilon/3$ since $y_n \in \Gamma(x_n)$. $d(y, y_n) \leq \epsilon/3$ since $y_n \to y$. $d(f(x), f(x_n)) \leq \epsilon/3$ since $f(x_n) \to f(x)$ by continuity of $f$.

Then; $d(y, f(x)) \leq d(y, y_n) + d(y_n, f(x_n)) + d(f(x_n), f(x)) \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

Thus $y \in \Gamma(x)$. So, $\Gamma$ is closed at $x$. Since $x$ was arbitrary, $\Gamma$ has a closed graph as we claimed.

9. (a) $\Gamma^{-1}(S) := \{x \in X : \Gamma(x) \subseteq S\}$: upper inverse image of $S$ under $\Gamma$.

Let $S \subseteq Y$ be open in $Y$.

Since $\Gamma$ is upperhemicontinuous, we have for all open $S$ in $Y$ with $\Gamma(x) \subseteq S$, there exists $\delta > 0$ s.t. $\Gamma(N_\delta(x)) \subseteq S$ for all $x \in X$.

Now take $x \in \Gamma^{-1}(S)$, that is, $\Gamma(x) \subseteq S$ by definition. Upperhemicontinuity implies that there exists $\delta > 0$ s.t. $\Gamma(N_\delta(x)) \subseteq S$. Need to show $N_\delta(x) \subseteq \Gamma^{-1}(S)$.
Let \( x' \in N_\delta(x) \). Then \( \Gamma(x') \subseteq S \).

Then by definition of \( \Gamma^{-1} \), we get \( x' \in \Gamma^{-1}(S) \). Thus \( N_\delta(x) \subseteq \Gamma^{-1}(S) \).

(b) \( \Gamma^{-1}(S) = \{ x \in X : \Gamma(x) \cap S \neq \emptyset \} \)

Let \( S \subseteq Y \) be open in \( Y \).

Since \( \Gamma \) is lower-hemicontinuous, we have for all open \( S \subseteq Y \) with \( \Gamma(x) \cap S \neq \emptyset \), there exists \( \delta > 0 \) such that \( \Gamma(x') \cap S \neq \emptyset \) for all \( x' \in N_\delta(x) \), for any \( x \in X \). Take \( x \in \Gamma^{-1}(S) \). Then \( \Gamma(x) \cap S \neq \emptyset \). So there exists \( \delta > 0 \) such that \( \Gamma(x') \cap S \neq \emptyset \) for all \( x' \in N_\delta(x) \).

Need to show \( N_\delta(x) \subseteq \Gamma^{-1}(S) \). Let \( x' \in N_\delta(x) \). Then we know \( \Gamma(x') \cap S \neq \emptyset \). Thus \( x' \in \Gamma^{-1}(S) \) by definition of \( \Gamma^{-1}(S) \). So \( N_\delta(x) \subseteq \Gamma^{-1}(S) \) and thus \( \Gamma^{-1}(S) \) is open in \( X \).

10. Assume that \( \Gamma \) is closed at some \( x \in X \) and \( Y \) is compact. Take any \((x^n)\) in \( X \) and \((y^n)\)in \( Y \) with \( \lim x_n = x \) and \( y^n \in \Gamma(x^n) \).

Since \( Y \) is compact every sequence in \( Y \) has a subsequence that converges to a point in \( Y \). Thus there exists \( (y^{nk}) \in Y \) s.t \( \lim y^{nk} = y \in Y \). So we get \( x^n \to x, y^n \to y \) and \( y^n \in \Gamma(x^n) \). By the closedness at \( x \), \( y \in \Gamma(x) \). Upperhemicontinuity at \( x \) follows directly from the proposition: \( \Gamma \) is upperhemicontinous at \( x \in X \) if for all \( (x_n) \in X, (y_n) \in Y \) with \( x_n \to x \) and \( y_n \in \Gamma(x_n) \) for all \( n \), there exists a subsequence \( y^{nk} \) (of \( y^n \)) that converges to a point in \( \Gamma(x) \).