1. Let \((X, d)\) be a metric space. Define \(d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}\) for any \(x, y \in X\).

(a) Show that \(d_1\) is a metric on \(X\).

(b) Show that if \(S \subseteq X\) is open in \((X, d)\), then \(S\) is open in \((X, d_1)\).

2. Let \((X, d)\) be a metric space. Let \(f : \mathbb{R}_+ \to \mathbb{R}\) be a concave and strictly increasing function with \(f(0) = 0\). Show that \((X, f \circ d)\) is a metric space.

3. Let \((X, d)\) be a metric space. Let \((x_n)\) and \((y_n)\) be two sequences in \(X\) with \(\lim x_n = x\) and \(\lim y_n = y\). Show that \(\lim d(x_n, y_n) = d(x, y)\).

4. Determine whether the following sets are open, closed or neither.

(a) \(\{(1/n, 1/n^2) : n \in \mathbb{N}\} \cup \{(0, 0)\} \subseteq \mathbb{R}^2\)

(b) \(\{(x, y, x^2y^2) : x^2 + y^2 < 1\} \subseteq \mathbb{R}^3\)

(c) \(\bigcup_{n=1}^{\infty} [-n, (n-1)/n] \subseteq \mathbb{R}\)

(d) \(\bigcap_{n=1}^{\infty} (0, 1/n] \subseteq \mathbb{R}\)

5. Let \((X, d)\) be a metric space. Let \(A, B \subseteq X\).

(a) Show that \(\text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B)\)

(b) Show that \(\text{Cl}(A) \cup \text{Cl}(B) = \text{Cl}(A \cup B)\)

6. Let \(A \subseteq \mathbb{R}\) be an open set and let \(B \subseteq \mathbb{R}\). Define \(AB = \{xy \in \mathbb{R} \mid x \in A, y \in B\}\) and for \(y \in \mathbb{R}\) define \(yA = \{xy \in \mathbb{R} \mid x \in A\}\). Is \(AB\) open in \(\mathbb{R}\)? Show that if \(y \neq 0\), then \(yA\) is open in \(\mathbb{R}\).

7. Let \((X, d)\) be a metric space and let \(S \subseteq X\). Show that \(x \in Bd(S)\) if and only if there exist \((x_n)\) in \(S\) and \((x'_n)\) in \(X \setminus S\) such that \(\lim x_n = x = \lim x'_n\).

8. Let \((X, d)\) be a metric space where \(d\) is the discrete metric.

(a) Show that any \(S \subseteq X\) is open in \((X, d)\).

(b) Show that any function \(f : X \to Y\) is continuous.

9. Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces, and let \(f : X \to Y\) be a continuous function. Show that \(f(\text{Cl}(A)) \subseteq \text{Cl}(f(A))\) for any \(A \subseteq X\).
10. Show that $f : (0, \infty) \to \mathbb{R}$ with $f(x) = 1/x$ is **not** uniformly continuous.

11. Let $(X, d)$ be a metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Show that if $f(x) = 0$ for all $x \in A \subseteq X$ then $f(x') = 0$ for all $x' \in \text{Cl}(A)$.

12. Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. Then, a function $f : X \to Y$ is called Lipschitz continuous if there exists a $K > 0$ such that $d_Y(f(x), f(y)) \leq Kd_X(x, y)$ for any $x, y \in X$. Show that if a function $f : X \to Y$ is Lipschitz continuous, then it is also uniformly continuous.

13. Let $f : [b, \infty) \to [b, \infty)$ be defined by $f(x) = \frac{1}{2}(x + \frac{a}{x})$ where $a > 0$ and $b > 0$. Is there a range of $a$ and $b$ values for which $f$ is a contraction mapping?

14. Let $(X, d)$ be a metric space and $\emptyset \neq S \subseteq X$. Suppose for all $x \in S$ there is a $\delta > 0$ such that $N_\delta(x) \subseteq S$. Show that if for all $\epsilon > 0$, $(N_\epsilon(y) \setminus \{y\}) \cap (X \setminus S) \neq \emptyset$, then $y \in X \setminus S$.

15. Let $(X, d)$ be a metric space and $A \subseteq X$. Show that if $A$ is bounded then there is a constant $M$ such that $d(x, y) \leq M$ for all $x, y \in A$.

16. Let $(X, d)$ be a metric space. Let $A, B \subseteq X$ with $A, B \neq \emptyset$. Show that $\text{Bd}(A) = [A \cap \text{Cl}(X \setminus A)] \cup [\text{Cl}(A) \setminus A]$.

17. Let $A \subseteq \mathbb{R}^n$ be a connected subset. Is $\mathbb{R}^n \setminus A$ also connected? Is $A$ open? Is $A$ closed?

18. Let $(X, d)$ be a metric space. Show that if $X$ is connected then for all nonempty proper subset $S$ of $X$, $\text{Bd}(S) \neq \emptyset$. Conversely show that if for all nonempty subset $S$ of $X$, $\text{Bd}(S) \neq \emptyset$, then $X$ is connected.

19. Let $(X, d)$ be a metric space and let $A, B \subseteq X$ be two connected subsets with $\text{Cl}(A) \cap B \neq \emptyset$. Show that $A \cup B$ is connected.

20. Let $(X, d)$ be a metric space and let $Y \subseteq X$. If $\text{Cl}(Y) = X$, then $Y$ is said to be dense in $X$. Let $f, g : X \to Y$ be two continuous functions where $Y$ is dense in $X$. Show that if $f(x) = g(x)$ for all $x \in Y$, then $f(x) = g(x)$ for all $x \in X$.