1. $Q = q_1^{1/3} q_2^{1/2}$

$$\pi = Q - p_1 q_1 - p_2 q_2$$

$$\frac{\partial \pi}{\partial q_1} = 0 : \frac{1}{3} q_1^{-2/3} q_2^{1/2} - p_1 = 0 : f_1(q; p) = 0$$

$$\frac{\partial \pi}{\partial q_2} = 0 : \frac{1}{2} q_1^{1/3} q_2^{-1/2} - p_2 = 0 : f_2(q; p) = 0$$

$$\det \begin{pmatrix} (-2/9)q_1^{-5/3} q_2^{1/2} & (1/6) q_1^{-2/3} q_2^{-1/2} \\ (1/6) q_1^{-2/3} q_2^{-1/2} & (-1/4) q_1^{1/3} q_2^{-3/2} \end{pmatrix} = q_1^{-4/3} q_2^{-1} \frac{2}{36} - \frac{1}{36} = (1/3) q_1^{-4/3} q_2^{-1} \neq 0.$$  

By IFT; $D_p q(p) = -D_q f(q, p)^{-1} D_p f(q; p)$ that is:

$$\begin{pmatrix} \frac{\partial q_1}{\partial p_1} & \frac{\partial q_1}{\partial p_2} \\ \frac{\partial q_2}{\partial p_1} & \frac{\partial q_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} (-2/9)q_1^{-5/3} q_2^{1/2} & (1/6) q_1^{-2/3} q_2^{-1/2} \\ (1/6) q_1^{-2/3} q_2^{-1/2} & (-1/4) q_1^{1/3} q_2^{-3/2} \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -9 q_1^{5/3} q_2^{-1/2} & -6 q_1^{2/3} q_2^{1/2} \\ -6 q_1^{2/3} q_2^{1/2} & -8 q_1^{-1/3} q_2^{3/2} \end{pmatrix}.$$

Thus, all partial derivatives are negative.

2. $u(c_1) + u(c_2)$ where $u' > 0$ and $u'' < 0$

$c_1 = w_1 - s$

$c_2 = w_2 + (1 + r) \theta s + \alpha$

tax rate on savings: $(1 - \theta)$

$\alpha = (1 - \theta)(1 + r)s$

$max u(c_1) + \beta u(c_2) \text{ s.t. } c_1 = w_1 - s \text{ and } c_2 = w_2 + (1 + r) \theta s + \alpha$

$max u(w_1 - s + \beta (w_2 + (1 + r) \theta s + \alpha))$

$-u'(w_1 - s) + \beta u'(w_2 + (1 + r) \theta s + \alpha)(1 + r)\theta = 0$

$\Rightarrow \beta u'(w_2 + (1 + r) \theta s + \alpha)(1 + r)\theta - u'(w_1 - s) = 0$

$\Rightarrow \beta u'(w_2 + (1 + r) \theta s + (1 - \theta)(1 + r)s)(1 + r)\theta - u'(w_1 - s) = 0$

$\Rightarrow \beta u'(w_2 + (1 + r)s)(1 + r)\theta - u'(w_1 - s) = 0 : f(s, \theta) = 0, s = s(\theta).$

By IFT;
\[
\frac{ds(\theta)}{d\theta} = -\frac{(\partial f)/(\partial \theta)}{(\partial f)/(\partial s)} = -\frac{\beta u'(w_2 + (1 + r)s)(1 + r)}{\beta u''(w_2 + (1 + r)s)(1 + r)^2 \theta + u''(w_1 - s)}
\]
\[
= -\frac{u'(w_2 + (1 + r)s)}{u''(w_2 + (1 + r)s)(1 + r)\theta + \frac{u''(w_1 - s)}{\beta(1 + r)}} > 0 \text{ since } u' > 0 \text{ and } u'' < 0.
\]

3. Denote the profit when price is \( p \) with \( \pi_H \), and the profit when price is \( p/2 \) with \( \pi_L \). Then, we have

\[
\begin{align*}
\pi_H &= py - c(y) - f \text{ if price is } p \text{ (has probability } q) \\
\pi_L &= \frac{py}{2} - c(y) - f \text{ if price is } p/2 \text{ (has probability } 1 - q) 
\end{align*}
\]

Expected utility is given by

\[
EU(\pi) = qu(\pi_H) + (1 - q)u(\pi_L)
\]

The firm’s problem is to choose \( y \) to maximize the expected utility from profit. That is,

\[
\max_y qu(py - c(y) - f) + (1 - q)u\left(\frac{py}{2} - c(y) - f\right)
\]

The first order condition with respect to \( y \) is then,

\[
q(p - c'(y))u'(py - c(y) - f) + (1 - q)(\frac{p}{2} - c'(y))u'(\frac{py}{2} - c(y) - f) = 0
\]

Define \( p^c = p \) to be the current price, or the certain price, from which there is no deviation if there is no uncertainty. But when there is uncertainty, it will stay the same with probability \( q \), and decrease to \( p/2 \) with probability \( 1 - q \). That is, certain situation is given by \( q = 1 \). So when there is no uncertainty the firm solves the following problem:

\[
\max_y u(py - c(y) - f)
\]

which has a first-order condition given by

\[
[p - c'(y)]u'(py - c(y) - f) = 0
\]

Since \( u'(.) > 0 \), this boils down to \( p = c'(y^c) \) where \( y^c \) is the optimal output of a price taker firm under certainty. This must be familiar, though. A perfectly competitive firm produces an output level which satisfies the equality between marginal cost and market price.
Now, define
\[ G(y; q) = q(p - c'(y))u'(py - c(y) - f) + (1 - q)(\frac{p}{2} - c'(y))u'(\frac{py}{2} - c(y) - f) = 0 \]

Then by implicit function theorem we have
\[ \frac{dy}{dq} = -\frac{\partial G(y; q)}{\partial q} \frac{\partial G(y; q)}{\partial y} \]

That is,
\[ \frac{dy}{dq} = -\frac{(p - c'(y))u'(\pi_H) - (\frac{p}{2} - c'(y))u'(\pi_L)}{q[(p - c'(y))^2u''(\pi_H) - c''(y)u'(\pi_H)] + (1 - q)((\frac{p}{2} - c'(y))^2u''(\pi_L) - c''(y)u'(\pi_L))} \]

The denominator is negative since it is the second order condition of the maximum. The expression in the numerator turns out to be positive. To see this, arrange the first-order condition as follows
\[
q(p - c'(y))u'(py - c(y) - f) + (1 - q)(\frac{p}{2} - c'(y))u'(\frac{py}{2} - c(y) - f) = 0
\]
\[ q(p - c'(y))u'(\pi_H) + (1 - q)(\frac{p}{2} - c'(y))u'(\pi_L) = 0 \]
\[ q[(p - c'(y))u'(\pi_H) - (\frac{p}{2} - c'(y))u'(\pi_L)] + (\frac{p}{2} - c'(y))u'(\pi_L) = 0 \]
\[ (p - c'(y))u'(\pi_H) - (\frac{p}{2} - c'(y))u'(\pi_L) = \frac{(\frac{p}{2} - c'(y))u'(\pi_L)}{q} \]

Hence \((p - c'(y))u'(\pi_H) - (\frac{p}{2} - c'(y))u'(\pi_L) > 0\) if and only if \(\frac{p}{2} < c'(y)\). So if \(\frac{p}{2} < c'(y^*)\) then the numerator is positive. Suppose the contrary, that is suppose \(\frac{p}{2} > c'(y^*)\). Then the numerator is negative and whole expression \(\frac{dy}{dq}\) becomes negative. This means that as \(q\) decreases, optimal \(y\) also increases. If optimal \(y\) is \(y^c\) whenever \(q = 1\), then for \(q < 1\), we get \(y^* > y^c\). Then \(c'(y^*) > c'(y^c)\) since \(c''(.) > 0\). Then \(\frac{p}{2} > c'(y^*) > c'(y^c) = p\). But this implies \(\frac{p}{2} > p\) which is a contradiction. Therefore it must be the case that \(\frac{p}{2} < c'(y^*)\), not \(\frac{p}{2} > c'(y^*)\)! This shows that the numerator is positive. Therefore, \(\frac{dy}{dq}\) turns out to be positive. This means that as \(q\) decreases, optimal \(y\) also decreases. Hence we conclude that \(y^* < y^c\), that is, whenever there is market price uncertainty, the firm produces less than it does when there is no uncertainty.

4. (a) Going through the Kuhn-Tucker calculation, at prices \(p^*\) and \(m^*\) the solution is,
\[ x^* = \left( \frac{m^*}{3p_1^*} \right) \left( \frac{2m^*}{3p_2^*} \right) \]
Thus, at general \((p, m)\), the solution function \(x^*\) is given by,
\[ x^*(p, m) = \left( \frac{m/(3p_1)}{(2m)/(3p_2)} \right). \] Also, \( \lambda_1^* = 1/m^* \). And the other KT multipliers (on the \( x \geq 0 \) constraint) equal zero. The value function is then determined by \( V(p, m) = f(x^*(p, m)) \), hence \( V(p, m) = \frac{1}{3} \ln \frac{m}{3p_1} + \frac{2}{3} \ln \frac{2m}{3p_2} = \ln(m) - \frac{1}{3} \ln(p_1) - \frac{2}{3} \ln(p_2) + \frac{1}{3} \ln \frac{1}{3} + \frac{2}{3} \ln \frac{2}{3} \).

Note that this makes some intuitive sense; in particular \( V \) is increasing in \( m \) and decreasing in \( p_1 \) and \( p_2 \). By direct calculation, at the point \((p^*, m^*)\),

\[
DV(p^*, m^*) = \begin{pmatrix}
\frac{\partial V}{\partial p_1}(p^*, m^*) \\
\frac{\partial V}{\partial p_2}(p^*, m^*) \\
\frac{\partial V}{\partial m}(p^*, m^*)
\end{pmatrix} = \begin{pmatrix}
-1/(3p_1^*) \\
-2/(3p_2^*) \\
1/m^*
\end{pmatrix}.
\]

(b) By the Envelope Theorem, since the parameters \( p \) and \( m \) don’t appear in the objective function, and with the first constraint written as \( g_1(x, p, m) = p.x - m \leq 0 \):

\[
DV(p^*, m^*) = \begin{pmatrix}
-\lambda_1^* x_1^* \\
-\lambda_1^* x_2^* \\
\lambda_1^*
\end{pmatrix}. \text{ Substitute in the value of } x^* \text{ and } \lambda^* \text{ found above and you will confirm that these two expressions for } DV(p^*, m^*) \text{ are indeed equal.}
\]

5. \( u(\theta) = \theta V(q(\theta)) - T(q(\theta)) = \max [\tilde{\theta} V(q(\tilde{\theta})) - T(q(\tilde{\theta}))]. \)

(a) Individual rationality constraints: \( \theta V(q(\theta)) - T(q(\theta)) \geq 0 \forall \theta. \)

Incentive compatibility constraints: \( \theta V(q(\theta)) - T(q(\theta)) \geq \theta V(q(\theta')) - T(q(\theta')) \forall \theta \theta'. \)

(b) By Envelope Thm.,

\[
\frac{dU}{d\theta} = U'(\theta) = V(q(\theta)).
\]

Integrating; \( U(\theta) = \int_{\theta}^{\bar{\theta}} V(q(u))du + U(\bar{\theta}) = \int_{\theta}^{\bar{\theta}} V(q(u))du \)

where \( U(\theta) = 0 \) since IR\( _\theta \) holds with equality, that is \( \theta v(q(\theta)) - T(q(\theta)) = 0. \)