1. Solve $\max_{x_1, x_2} 6x_1 + 2x_1x_2 - 2x_1^2 - 2x_2^2$ subject to $x_1 + 2x_2 - 2 \leq 0$ and $1 + x_1 - x_2^2 \geq 0$.

2. Let there be two goods $x$ and $y$ with prices $p_x > 0$ and $p_y > 0$, respectively. Let the income of a consumer be given by $I > 0$ and her utility function be given by $u(x, y) = y + \alpha \ln(x)$ with $\alpha > 0$. Solve her utility maximization problem for $x, y \in \mathbb{R}_+$.

3. Consider the cost minimization problem of a firm who is producing (at least) $y > 0$ units. Let $F(y) = \{x \in \mathbb{R}_+^n : g(x) \geq y\}$ be the feasible set. Let $w \in \mathbb{R}_+^n$ be the input price vector. Thus the problem is $\min_x w \cdot x$ over the set $F(y)$. Solve this problem for $n = 2$ and $g(x_1, x_2) = x_1^2 + x_2^2$.


5. Let $S \subseteq \mathbb{R}^n$ be a convex set, and let $f : S \rightarrow \mathbb{R}$ be a function.

   (a) If $f$ is concave, show that the set $\arg\max\{f(x) : x \in S\}$ of maximizers of $f$ on $S$ is either empty or convex.

   (b) If $f$ is strictly concave, show that the set $\arg\max\{f(x) : x \in S\}$ of maximizers of $f$ on $S$ is either empty or singleton.

6. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Given any $x$ and $h$ in $\mathbb{R}^n$, define $g_{x,h}(\cdot)$ by $g_{x,h}(t) = f(x + th)$, $t \in \mathbb{R}$. Then, show that, $f$ is concave on $\mathbb{R}^n$ if and only if the function $g_{x,h}(\cdot)$ is concave in $t$ for each fixed $x, h \in \mathbb{R}^n$.

7. Let $f : C \rightarrow \mathbb{R}$ where $C \subseteq \mathbb{R}^n$ is convex. The upper-contour set of $f$ at $y \in \mathbb{R}$ is the set $U_f(y) = \{x \in C : f(x) \geq y\}$. Show that $f$ is quasi-concave if and only if $U_f(y)$ is convex for all $y \in \mathbb{R}$.

8. Let $C \subseteq \mathbb{R}^n$ be a convex set, and let $f : C \rightarrow \mathbb{R}$ be a function.

   (a) Show that the set $\arg\max\{f(x) : x \in C\}$ of maximizers of $f$ on $S$ is either empty or convex, if $f$ is concave.

   (b) Show that the set $\arg\max\{f(x) : x \in C\}$ of maximizers of $f$ on $C$ is either empty or singleton, if $f$ is strictly concave.

   (c) Suppose the set $\arg\max\{f(x) : x \in C\}$ of maximizers of $f$ on $C$ is singleton. Is $f$ strictly concave? Prove or give a counterexample.
9. Suppose that there is a firm producing the output $y$ using two inputs $x_1$ and $x_2$ in non-negative quantities through the production technology given by $y = f(x_1, x_2) = x_1^{1/4} x_2^{1/4}$. The firm earns a price of $p > 0$ for each unit of $y$ it sells. The firm has an inventory of $k_1 > 0$ units of the input $x_1$ and $k_2 > 0$ units of the input $x_2$. More units of $x_1$ and $x_2$ can be purchased from the market at the unit prices $p_1 > 0$ and $p_2 > 0$, respectively. Also, the firm can sell any unused amount of its inputs to the market at these prices.

(a) Describe the firm’s profit maximization problem and characterize the critical points.

(b) Find the critical point for $p = p_1 = p_2 = 1, k_1 = 4, k_2 = 2$. Is it a global maximum?

10. (a) Define Kuhn-Tucker-Karush Point for a constrained maximization problem.

(b) Consider a firm whose objective is to maximize revenue without letting the profit drop below some fixed level, $\bar{\pi}$. There is an advertising cost $a \in \mathbb{R}_+$. Let $R(q, a)$ denote the firm’s revenue when the level of production is $q \in \mathbb{R}_+$ and the advertising cost is $a \in \mathbb{R}_+$. Let $C(q)$ denote the cost of producing $q$ units. Assume that $C$ and $R$ are both continuously differentiable with $C' > 0$ and $\partial R/\partial a > 0$. Firm maximizes $R(q, a)$ subject to $\pi = R(q, a) - C(q) - a \geq \bar{\pi}$ and $q \geq 0$ and $a \geq 0$. Characterize the KTK points $(q^*, a^*)$ with $q^* > 0$, and show that, at any such KTK point, (i) the profit realized is $\bar{\pi}$, and (ii) $q^*$ is greater than the profit maximizing output.

11. Consider the following problem of maximizing $x^2 + x + 4y^2$ subject to the constraints $2x + 2y \leq 1$, $x \geq 0$ and $y \geq 0$. Find the KTK points.