1. Define \( L(x_1, x_2, \lambda_1, \lambda_2) = 6x_1 + 2x_1x_2 - 2x_1^2 - 2x_2^2 + \lambda_1(2 - x_1 - 2x_2) + \lambda_2(1 + x_1 - x_2^2) \).

Then, \( D_{x_1}L = 6 + 2x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0, \ D_{x_2}L = 2x_1 - 4x_2 - 2\lambda_1 - 2\lambda_2x_2 = 0 \) \hspace{1cm} (1)

\[ 2 - x_1 - 2x_2 \geq 0, \ \lambda_1 \geq 0, \ \lambda_1(2 - x_1 - 2x_2) = 0 \hspace{1cm} (2) \]

\[ 1 + x_1 - x_2^2 \geq 0, \ \lambda_2 \geq 0, \ \lambda_2(1 + x_1 - x_2^2) = 0 \hspace{1cm} (3) \]

(case1) Suppose \( \lambda_1 > 0, \ \lambda_2 > 0 \). Then, \( 2 - x_1 - 2x_2 = 0 \) and \( 1 + x_1 - x_2^2 = 0 \), which imply \( x_1 = 0, x_2 = 1 \). Then, (1) implies \( 8 - \lambda_1 + \lambda_2 = 0 \) and \( -4 - 2\lambda_1 - 2\lambda_2 = 0 \). Solving these two together we get, \( \lambda_1 = 3 \) and \( \lambda_2 = -5 \). This cannot be optimum since \( \lambda_2 \geq 0 \). So, at least one of the constraints must be non binding.

(case2) Suppose \( 1 + x_1 - x_2^2 > 0 \) and \( 2 - x_1 - 2x_2 = 0 \). Then, \( \lambda_2 = 0 \). Note, \( \text{rank}(D[2 - x_1 - 2x_2]) = \text{rank}([-1 - 2]) = 1 \leq 2 \). Then;

\[ D_{x_1}L = 6 + 2x_2 - 4x_1 - \lambda_1 = 0, \ D_{x_2}L = 2x_1 - 4x_2 - 2\lambda_1 = 0, \ 2 - x_1 - 2x_2 = 0 \]

which implies \( x_1 = 10/7, x_2 = 2/7, \lambda_1 = 6/7 \). With these values, check if \( g_2(x) > 0 \) and it holds \( 1 + x_1 - x_2^2 > 0 \).

(case3) Suppose \( 2 - x_1 - 2x_2 > 0 \) and \( 1 + x_1 - x_2^2 = 0 \). Then, \( \lambda_1 = 0 \). Note, \( \text{rank}(D[1 + x_1 - x_2^2]) = \text{rank}([1 - 2x_2]) = 1 \leq 2 \). Then;

\[ D_{x_1}L = 6 + 2x_2 - 4x_1 + \lambda_2 = 0, \ D_{x_2}L = 2x_1 - 4x_2 - 2\lambda_2x_2 = 0, \ 1 + x_1 - x_2^2 = 0 \]

which has 3 solutions where each solution has \( \lambda_2 < 0 \). Therefore this case can also be ruled out.

(case4) Suppose \( 2 - x_1 - 2x_2 > 0 \) and \( 1 + x_1 - x_2^2 > 0 \). Then, \( \lambda_1 = \lambda_1 = 0 \). Then,

\[ D_{x_1}L = 6 + 2x_2 - 4x_1 = 0, \ D_{x_2}L = 2x_1 - 4x_2 = 0 \]

which gives \( x_1 = 2 \) and \( x_2 = 1 \). But then \( 2 - x_1 - 2x_2 = -2 < 0 \), contradicting \( 2 - x_1 - 2x_2 > 0 \). Thus this case is also ruled out.

Therefore, we conclude that the only possible solution is \( x_1^* = 10/7, x_2^* = 2/7 \) with \( \lambda_1^* = 6/7 \) and \( \lambda_2^* = 0 \). In fact this is the unique global solution!

2. Define \( L = y + \alpha \ln(x) + \lambda(I - px - py) + \lambda_x x + \lambda_y y \). We have,

\[ D_x L = \frac{\alpha}{x} - \lambda p_x + \lambda_x = 0 \quad \text{and} \quad D_y L = 1 - \lambda p_y + \lambda_y = 0, \]

\( I - px - py \geq 0 \); \( \lambda \geq 0 \); \( \lambda(I - px - py) = 0 \) and

\[ x \geq 0 \; ; \; y \geq 0 ; \; \lambda_x \geq 0 ; \; \lambda_y \geq 0 ; \; x\lambda_x = 0 ; \; y\lambda_y = 0. \]
Suppose $\mathcal{I} > p_x x + p_y y$. Then $\lambda = 0$. Then $\frac{\alpha}{x} = -\lambda x$ which cannot happen since $\alpha > 0$. So $\mathcal{I} = p_x x + p_y y$, and $\lambda \geq 0$.

Need to check $x = 0, y = 0, x > 0, y > 0$.

(case1) If $x = y = 0$ then $\mathcal{I} - p_x x - p_y y = 0$ implies $\mathcal{I} = 0$!

(case2) If $x = 0, y > 0$ then $y = \mathcal{I}/p_y >$. Also, $\lambda = 1/p_y$ since $\lambda y = 0$. Then $\frac{\alpha}{x} = \frac{p_x}{p_y} - \lambda x$ which cannot happen!

(case3) If $x > 0, y = 0$ then $x = \mathcal{I}/p_x$. Also, since $\lambda x = 0$, we have $\lambda = \alpha/\mathcal{I}$. Then $1 = (\frac{\alpha}{\mathcal{I}}) p_y - \lambda y$, that is, $\lambda y = (\frac{\alpha p_y}{\mathcal{I}}) - 1$ So this case is valid only for $\alpha p_y \geq \mathcal{I}$

(case4) If $x > 0, y > 0$ then $\lambda x = \lambda y = 0$. This implies $\frac{\alpha}{p_x x} = \lambda = \frac{1}{p_y}$, that is, $x = \frac{\alpha p_y}{p_x}$.

Then, $\mathcal{I} = p_x \frac{\alpha p_y}{p_x} + p_y y = (\alpha + y)p_y$. Hence, $y = \frac{\mathcal{I}}{p_y} - \alpha$ which could be the case only if $\mathcal{I} > \alpha p_y$, since $y > 0$.

If we split the space of all possible values of $\{\alpha, p_x, p_y, \mathcal{I}\}$ into two sets:

$\{(\mathcal{I}, p_x, p_y): \mathcal{I} \leq \alpha p_y\}$ and $\{(\mathcal{I}, p_x, p_y): \mathcal{I} > \alpha p_y\}$ we get

If $\mathcal{I} \leq \alpha p_y$ then $x^* = \mathcal{I}/p_x$ and $y^* = 0$.

If $\mathcal{I} > \alpha p_y$ then $x^* = \alpha p_y/p_x$ and $y^* = (\mathcal{I}/p_y) - \alpha$.

3. The constraint set of this problem is defined through three inequality constraints,

$h_1(x_1, x_2) = x_1 \geq 0$
$h_2(x_1, x_2) = x_2 \geq 0$
$h_3(x_1, x_2) = x_1^2 + x_2^2 - y \geq 0$.

Assume that all the parameters of the problem $w_1, w_2$ and $y$ are strictly positive. To check the constraint qualification will hold at the optimum, we first identify all possible combinations of the constraints that can, in principle, hold with equality at the optimum. Since there are three constraints, there are eight cases to be checked: $\emptyset, h_1, h_2, h_3, (h_1, h_2), (h_1, h_3), (h_2, h_3)$ and $(h_1, h_2, h_3)$—where $(h_1, h_3)$ means for instance that the first and the third ones bind and the second does not. Of these, the last can be ruled out since $h_1 = h_2 = 0$ implies $x_1^2 + x_2^2 = 0$, whereas the constraint set requires $x_1^2 + x_2^2 \geq y$. It is also apparent that since $w_1$ and $w_2$ are strictly positive, we must have $h_3 = 0$ at the optimum (i.e., total production $x_1^2 + x_2^2$ must exactly equal $y$), or costs could be reduced by reducing output. This means there are only three possible cases at the optimum: $(h_3), (h_1, h_3)$ and $(h_2, h_3)$. We will show that in each case, the constraint qualification must hold.

For $(h_1, h_3)$, since $h_1$ and $h_3$ hold with equality, we have $x_1 = 0$ and $x_1^2 + x_2^2 = y$, so $x_2 = \sqrt{y}$. Therefore,
\[ D(h_1, h_3)(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 2x_1 & 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2\sqrt{y} \end{pmatrix} \]

Since this matrix evidently has the required rank, it follows that if the optimum occurs at a point where \( h_1 \) and \( h_3 \) are the only binding constraints, the constraint qualification will be met.

For \((h_2, h_3)\), an identical argument, with the obvious changes, shows that the constraint qualification is not a problem if the optimum happens to occur at a point where \( h_2 \) and \( h_3 \) are binding. This leaves the case \( h_3 = 0 \) where we have \( Dh_3(x_1, x_2) = [2x_1 \ 2x_2] \).

Since we are assuming that only \( h_3 \) holds with equality, we must have \( x_1, x_2 > 0 \), so \( \text{rank}(Dh_3(x_1, x_2)) = 1 \) as required.

Summing up, the optimum must occur at a point where \( h_3 = 0 \), but no matter where on this set the optimum occurs, the constraint qualification will be met. It follows that the set of critical points of the lagrangean must contain the solution(s) of the problem.

The Lagrangean \( \mathcal{L} \) in this problem has the form
\[ \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = -w_1x_1 - w_2x_2 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3(x_1^2 + x_2^2 - y). \]

Note that we have implicitly set the problem up as a maximization problem, with objective \(-w_1x_1 - w_2x_2\). The critical points of \( \mathcal{L} \) are the solutions \((x_1, x_2, \lambda_1, \lambda_2, \lambda_3)\) to the following system of equations:

1. \(-w_1 + \lambda_1 + 2\lambda_3x_1 = 0.\)
2. \(-w_2 + \lambda_2 + 2\lambda_3x_2 = 0.\)
3. \(\lambda_1 \geq 0, x_1 \geq 0, \lambda_1x_1 = 0.\)
4. \(\lambda_2 \geq 0, x_2 \geq 0, \lambda_2x_2 = 0.\)
5. \(\lambda_3 \geq 0, x_1^2 + x_2^2 - y \geq 0, \lambda_3(x_1^2 + x_2^2 - y) = 0.\)

We fix a subset \( \mathcal{C} \) of \( \{h_1, h_2, h_3\} \) and find the set of all possible solutions to these equations when only the constraints in \( \mathcal{C} \) hold with equality. As \( \mathcal{C} \) ranges over all possible subsets of \( \{h_1, h_2, h_3\} \), we obtain the set of all critical points of \( \mathcal{L} \).

Once again, this process is simplified by the fact that we do not have to consider all possible subsets \( \mathcal{C} \). First, as we have mentioned, \( h_3 \) must be effective at an optimum, so it suffices to find all critical points of \( \mathcal{L} \) at which \( h_3 = 0 \). Second, the case \( \mathcal{C} = \{h_1, h_2, h_3\} \) is ruled out, since \( h_1 = h_2 = 0 \) violates the third constraint that \( h_3 \geq 0 \). This results in three possible values for \( \mathcal{C} \), namely, \( \mathcal{C} = \{h_3\}, \mathcal{C} = \{h_2, h_3\}, \) and \( \mathcal{C} = \{h_1, h_3\} \). We consider each of these in turn.

**Case1: \( \mathcal{C} = \{h_3\} \)**

Since we must have \( x_1 > 0 \) and \( x_2 > 0 \) in this case, the complementary slackness conditions imply \( \lambda_1 = \lambda_2 = 0 \). Substituting these values into the first two equations that define the
critical points of $L$, we obtain $2\lambda_3 x_1 = w_1$ and $2\lambda_3 x_2 = w_2$, which imply $x_1 = \frac{w_1}{w_2} x_2$. By hypothesis, we also have $h_3 = 0$ or $x_1^2 + x_2^2 = y$. Therefore, we have $x_1 = \frac{(w_2 y)}{(w_1^2 + w_2^2)^{1/2}}$, $x_2 = \frac{(w_2 y)}{(w_1^2 + w_2^2)^{1/2}}$, $x_3 = \frac{(w_1^2 + w_2^2)^{1/2}}{y}$. Combined with $\lambda_1 = \lambda_2 = 0$, this represents the unique critical point of $L$ in which $h_1 > 0, h_2 > 0$ and $h_3 = 0$. Note that the value of the objective function at this critical point is $-w_1 x_1 - w_2 x_2 = -(w_1^2 + w_2^2)^{1/2} y^{1/2}$.

**Case 2:** $C = \{h_2, h_3\}$

In this case, we have $x_1 > 0$. Therefore, we must also have $\lambda_1 = 0$. Substituting $x_1 = \sqrt{y}$ and $\lambda_1 = 0$ into the first of the equations that define the critical points of $L$, we obtain $2\lambda_3 x_1 - w_1 = 2\lambda_3 \sqrt{y} - w_1 = 0$, or $\lambda_3 = w_1 / 2\sqrt{y}$. Finally, substituting $x_2 = 0$ into the second equation defining the critical points of $L$, we get $\lambda_2 = w_2$. Thus, the unique critical point of $L$ which satisfies $h_2 = h_3 = 0$ is $(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = (\sqrt{y}, 0, 0, w_2, w_1 / 2\sqrt{y})$. The value of the objective function at this critical point is $-w_1 y^{1/2}$.

**Case 3:** $C = \{h_1, h_3\}$

This is the same as Case 2, with the obvious changes. The unique critical point of $L$ that falls in this case is $(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = (0, \sqrt{y}, w_1, 0, w_2 / \sqrt{y})$. The value of the objective function at this critical point is $w_2 y^{1/2}$.

Summing up, $L$ has three critical points, and the values taken by the objective function at these three points are $-(w_1^2 + w_2^2)^{1/2} y^{1/2}, -w_1 y^{1/2}$ and $-w_2 y^{1/2}$. Now, we have already established that the set of critical points of $L$ in this problem contains the solution(s) to the problem; and, therefore, that the point(s) that maximize the value of the objective function among the set of critical points must be the solution(s) to the problem. All that remains now is to compare the value of the objective function at the three critical points.

Since $w_1 > 0$ and $w_2 > 0$, it is always the case that $(w_1^2 + w_2^2)^{1/2} > w_1$ and, therefore, that $-(w_1^2 + w_2^2)^{1/2} y^{1/2} < -w_1 y^{1/2}$.

We may, as a consequence, ignore the first value of the objective function, and the point that it represents. Comparing the value of the objective function at the remaining two points, it can be seen that

- When $w_1 < w_2$, then the larger of the two values is $-w_1 y^{1/2}$, which arises at $(x_1, x_2) = (y^{1/2}, 0)$. Therefore, the problem has a unique solution when $w_1 < w_2$, namely $(y^{1/2}, 0)$.
- When $w_1 < w_2$, then the larger of the two values is $-w_1 y^{1/2}$, which arises at $(x_1, x_2) = (0, y^{1/2})$. Therefore, the problem has a unique solution when $w_1 > w_2$, namely $(0, y^{1/2})$.
- When $w_1 = w_2 = w > 0$, then these two values of the objective function coincide. Therefore, if $w_1 = w_2$, the problem has two solutions, namely $(y^{1/2}, 0)$ and $(0, y^{1/2})$. 


4. Recall the theorem (Theorem 5.5 in Vohra page 91) that states: For \( A_{m \times n}, b \in \mathbb{R}^m \), \( f \) continuous and differentiable, if \( x^* \) is a local maximum for \( \max \{ f(x) : Ax \geq b \} \), then there exist a nonzero \( \lambda \in \mathbb{R}^n_+ \) such that \( \nabla f(x) + A^T \lambda = 0 \) and \( \lambda (A x^* - b) = 0 \). Applying this theorem to the optimization problem at hand, let \( f(x) = -(1/2)x Cx - px \). Then, since this \( f(\cdot) \) is continuous and differentiable, we can apply the theorem: For a local maximum \( x^* \), there exist a nonzero \( \lambda \in \mathbb{R}^m_+ \) such that \( \nabla f(x) + A^T \lambda = 0 \) and \( \lambda (A x^* - b) = 0 \). The former is equivalent to \(-Cx^* + p + A^T \lambda = 0\), or \( Cx^* + p = A^T \lambda = \sum_{j=1}^m \lambda_j a_j \) where \( a_j \) is the \( j^{th} \) row of \( A \). Since \( \lambda (A x^* - b) = 0 \), for those \( a_i \) with the property \( a_i x^* = b_i \), it must be \( \lambda_i = 0 \). So, \( Cx^* + p = \sum_{i \in I} \lambda_i a_i \) where \( I = \{ i : a_i x^* = b_i \} \). Pick \( w_i = \lambda_i \).

For the sufficiency part, we need to show that \( f(x) = -(1/2)x Cx - px \) is concave, or the Hessian of \( f(\cdot) \) is negative semidefinite, or the Hessian of \(-f(\cdot) = (1/2)x Cx + px \) is positive semidefinite. But it is easy to see that \( H_{-f(\cdot)} = C \) (the second derivatives of \( p \cdot x \) vanishes). Since \( C \) is positive semidefinite the function \(-f(\cdot) \) is convex (convexity of a function is equivalent to the positive semidefiniteness of the Hessian of the function). Thus \( f(\cdot) \) is concave, and hence the necessity condition is now also sufficient.

5. (a) Suppose \( x_1 \) and \( x_2 \) are both maximizers of \( f \) on \( D \). Then, we have \( f(x_1) = f(x_2) \). Further, for \( \lambda \in (0, 1) \), we have \( f(\lambda x_1 + (1 - \lambda) x_2) \geq \lambda f(x_1) + (1 - \lambda) f(x_2) = f(x_1) \), and this must hold with equality or \( x_1 \) and \( x_2 \) would not be maximizers. Thus, the set of maximizers must be convex, completing the proof.

(b) Suppose \( \arg \max \{ f(x) : x \in D \} \) is nonempty. We will show it must contain a single point. By part (a) above \( \arg \max \{ f(x) : x \in D \} \) must be convex. Suppose this set contains two distinct points \( x \) and \( y \). Pick and \( \lambda \in (0, 1) \) and let \( z = \lambda x + (1 - \lambda) y \). Then, \( z \) must also be a maximizer of \( f \), so we must have \( f(z) = f(x) = f(y) \). However, by strict concavity of \( f \), \( f(z) = f(\lambda x + (1 - \lambda) y) > \lambda f(x) + (1 - \lambda) f(y) = f(x) \), a contradiction.

6. (⇒) First suppose that \( f \) is concave on \( \mathbb{R}^n \). Fix any \( x \) and \( h \) in \( \mathbb{R}^n \). For any pair of real numbers \( t \) and \( t' \), and any \( \alpha \in (0, 1) \), we have,
\[
g_{x,h}(\alpha t + (1 - \alpha) t') = f(x + \alpha th + (1 - \alpha) t'h) = f(\alpha(x + th) + (1 - \alpha)(x + t'h)) \geq \alpha f(x + th) + (1 - \alpha) f(x + t'h) = \alpha g_{x,h}(t) + (1 - \alpha) g_{x,h}(t').
\]
So \( g_{x,h}(\cdot) \) is concave in \( t \).

(⇐) Next, suppose for any \( x \), \( h \) in \( \mathbb{R}^n \), \( g_{x,h}(\cdot) \) is concave in \( t \). Pick any \( z_1, z_2 \in \mathbb{R}^n \) and any \( \alpha \in (0, 1) \). Let \( z(\alpha) = \alpha z_1 + (1 - \alpha) z_2 \). In order to prove that \( f \) is concave, we are required to show that \( f(z(\alpha)) \geq \alpha f(z_1) + (1 - \alpha) f(z_2) \) for any \( \alpha \in (0, 1) \). Consider the function \( g_{x,h}(\cdot) \) where \( x = z_1 \) and \( h = z_2 - z_1 \). Note that \( g_{x,h}(0) = f(z_1) \) and \( g_{x,h}(1) = f(z_2) \). Moreover, \( g_{x,h}(\alpha) = f(z_1 + \alpha(z_2 - z_1)) = f((1 - \alpha)z_1 + \alpha z_2) \). Since \( g_{x,h}(\cdot) \) is concave by hypothesis, for any \( \alpha \in (0, 1) \), we have \( f((1 - \alpha)z_1 + \alpha z_2) = g_{x,h}(\alpha) = g_{x,h}((1 - \alpha)0 + \alpha 1) \geq (1 - \alpha)g_{x,h}(0) + \alpha g_{x,h}(1) = (1 - \alpha) f(z_1) + \alpha f(z_2) \), which completes the proof that \( f \) is concave.
7. \(\Rightarrow\) First, suppose that \(f\) is quasi-concave, that is, \(f[\lambda x + (1 - \lambda)y] \geq \min\{f(x), f(y)\}\) for all \(x, y \in D\) and for all \(\lambda \in (0,1)\). Let \(a \in \mathbb{R}\). If \(U_f(a)\) is empty or contains only one point, it is evidently convex, so suppose it contains at least two points \(x\) and \(y\). Then \(f(x) \geq a\) and \(f(y) \geq a\), so \(\min\{f(x), f(y)\} \geq a\). Now, for any \(\lambda \in (0,1)\), we have \(f[\lambda x + (1 - \lambda)y] \geq \min\{f(x), f(y)\}\) by hypothesis, and so \(\lambda x + (1 - \lambda)y \in U_f(a)\). Since \(a\) was arbitrary, the proof is complete.

\(\Leftarrow\) Now, suppose that \(U_f(a)\) is a convex set for each \(a \in \mathbb{R}\). Let \(x, y \in D\) and \(\lambda \in (0,1)\). Assume, without loss of generality, that \(f(x) \geq f(y)\). Letting \(f(y) = a\), we have \(x, y \in U_f(a)\). By the convexity of \(U_f(a)\), we have \(\lambda x + (1 - \lambda)y \in U_f(a)\), which means \(f[\lambda x + (1 - \lambda)y] \geq f(y) \geq a = f(y) = \min\{f(x), f(y)\}\), proving that \(f\) is quasi-concave.

8. (a) Suppose \(x_1\) and \(x_2\) are both maximizers of \(f\) on \(D\). Then, we have \(f(x_1) = f(x_2)\). Further, for \(\lambda \in (0,1)\), we have \(f[\lambda x_1 + (1 - \lambda)x_2] \geq \lambda f(x_1) + (1 - \lambda)f(x_2) = f(x_1)\), and this must hold with equality or \(x_1\) and \(x_2\) would not be maximizers. Thus, the set of maximizers must be convex, completing the proof.

(b) Suppose \(\arg \max\{f(x) : x \in D\}\) is nonempty. We will show it must contain a single point. By part (a) above \(\arg \max\{f(x) : x \in D\}\) must be convex. Suppose this set contains two distinct points \(x\) and \(y\). Pick and \(\lambda \in (0,1)\) and let \(z = \lambda x + (1 - \lambda)y\). Then, \(z\) must also be a maximizer of \(f\), so we must have \(f(z) = f(x) = f(y)\). However, by strict concavity of \(f\), \(f(z) = f[\lambda x + (1 - \lambda)y] > f(x) + (1 - \lambda)f(y) = f(x)\), a contradiction.

(c) No. Take a convex function on \([0,1]\) with a unique maximum at \(x = 1\) for instance.

9. (a) The profit maximization problem can be described as

\[
\max_{x_1, x_2} p x_1^{1/4} x_2^{1/4} - p_1(x_1 - k_1) - p_2(x_2 - k_2) \quad \text{s.t.} \quad x_1 \geq -k_1 \text{ and } x_2 \geq -k_2.
\]

The critical points are the solutions \((x_1, x_2, \lambda_1, \lambda_2)\) to the following system:

\[
\begin{align*}
(1) & \quad (1/4)p x_1^{-3/4} x_2^{1/4} - p_1 + \lambda_1 = 0, \\
(2) & \quad (1/4)p x_1^{1/4} x_2^{-3/4} - p_2 + \lambda_2 = 0, \\
(3) & \quad \lambda_1 \geq 0, x_1 + k_1 \geq 0, \lambda_1(x_1 + k_1) = 0, \\
(4) & \quad \lambda_2 \geq 0, x_2 + k_2 \geq 0, \lambda_2(x_2 + k_2) = 0.
\end{align*}
\]

Note that whenever \(x_i = -k_i\) for a \(k_i > 0\), then the expression \((-k_i)^{-3/4}\) is a complex number. So, \(x_i = -k_i\) is not possible. So it must be the case that \(x_i + k_i > 0\) for both \(i = 1, 2\). This means that \(\lambda_i = 0\), for \(i = 1, 2\). Then, we have (1) \((1/4)p x_1^{-3/4} x_2^{1/4} - p_1 = 0\), and (2) \((1/4)p x_1^{1/4} x_2^{-3/4} - p_2 = 0\), which together imply \(p_1 x_1 = p_2 x_2\). Plugging \(x_2 = p_1 x_1/p_2\) into (1) we get \((1/4)p x_1^{-3/4}(p_1 x_1/p_2)^{1/4} - p_1 = 0\) which implies \(x_1 = \frac{(p/4)^2}{p_1^{3/4} p_2^{1/4}}\) and then solving for \(x_2\) we get \(x_2 = \frac{p_1^{3/4} p_2^{1/4}}{p_1^{3/4} p_2^{1/4}}\), as the (only) critical point.
(b) For these parameter values we get \( x_1 = x_2 = 1/16 \) as the critical point. This is actually the global maximum because the Hessian matrix of \( x_1^{1/4}x_2^{1/4} - x_1 - x_2 + 6 \) is negative semidefinite. To see this note that for any \( u \in \mathbb{R}^2 \), 
\[
\begin{bmatrix}
-3(1/16)x_1^{-7/4}x_2^{1/4} & (1/16)x_1^{-3/4}x_2^{-3/4} \\
(1/16)x_1^{-3/4}x_2^{-3/4} & -(3/16)x_1^{1/4}x_2^{-7/4}
\end{bmatrix}
\]
\[
(-1/16)x_1^{-7/4}x_2^{-7/4}(3u_1^2x_2^2 - 2u_1u_2x_1x_2 + 3u_2^2x_1^2) \leq 0 
\] for any \( x_1, x_2 \in \mathbb{R} \).

10. (a) A point \( x^* \) is called a Kuhn-Tucker-Karush point for a constrained problem 
\[
\max_{x \in \mathbb{R}^n} \{ f(x) : f^i(x) \geq 0 \ \forall i \in M \} \text{ if } \exists \{ \lambda_i \}_{i \in M} \text{ s.t.}
\]

- (1) \( \nabla f(x^*) + \sum_{i \in M} \lambda_i \nabla f^i(x^*) = 0 \),
- (2) \( \lambda_i f^i(x^*) = 0, \forall i \in M \),
- (3) \( \lambda_i \geq 0, \forall i \in M \),
- (4) \( f^i(x^*) \geq 0, \forall i \in M \).

(b) The Lagrangian is \( L(q, a; \lambda_1, \lambda_2) = R(q, a) + \lambda_1 a - \lambda_2 [\bar{\pi} - R(q, a) + C(q) + a] \). First order conditions are

- \[
\frac{\partial L}{\partial q} = (1 + \lambda_2) \frac{\partial R}{\partial q} - \lambda_2 C'(q) = 0 \quad (4)
\]
- \[
\frac{\partial L}{\partial a} = (1 + \lambda_2) \frac{\partial R}{\partial a} + \lambda_1 - \lambda_2 = 0 \quad (5)
\]

And we have, \( \lambda_1 a = 0 \) and \( \lambda_2 [\bar{\pi} - R(q, a) + C(q) + a] = 0 \) and \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) and \( a \geq 0 \) and \( \bar{\pi} - R(q, a) + C(q) + a \geq 0 \). Any \( (q > 0, a) \) that satisfy these is a KTK point.

(i) Since \( \partial R/\partial a > 0 \) and \( \lambda_2 \geq 0 \), we get \( \lambda_1 - \lambda_2 < 0 \) by (2) above. Then, since \( \lambda_1 \geq 0 \) we get \( \lambda_2 > 0 \). Thus, \( \bar{\pi} = R(q, a) - C(q) - a \) at any KTK point (with \( q > 0 \)), that is, the profit realized is \( \bar{\pi} \).

(ii) By (1) above, we have at a KTK \((q, a)\),

\[
(1 + \lambda_2) \frac{\partial \bar{\pi}}{\partial q} = (1 + \lambda_2) \left[ \frac{\partial R}{\partial q} - C'(q) \right] = \frac{\partial L}{\partial q} - C'(q) = 0 - C'(q) < 0
\]

Thus, since \( \frac{\partial \bar{\pi}}{\partial q} < 0 \) at a KTK point, the output in KTK point is greater than the profit maximizing quantity.

11. Lagrangian is \( L(x, y, \lambda_1, \lambda_2, \lambda_3) = x^2 + x + 4y^2 - \lambda_1 (2x + 2y - 1) + \lambda_2 x + \lambda_3 y \)

Then the KTK points are characterised by

- \[
\frac{\partial L}{\partial x} = 2x + 1 - 2\lambda_1 + \lambda_2 = 0 \quad (6)
\]
- \[
\frac{\partial L}{\partial y} = 8y - 2\lambda_1 + \lambda_3 = 0 \quad (7)
\]
\[ \lambda_1(2x + 2y - 1) = 0 \quad (8) \]
\[ \lambda_2x = 0 \quad (9) \]
\[ \lambda_3y = 0 \quad (10) \]
\[ \lambda_1, \lambda_2, \lambda_3 \geq 0 \quad (11) \]
\[ 2x + 2y - 1 \leq 0, \quad x \geq 0, \quad y \geq 0 \quad (12) \]

(6) can be written as \[ 2x + 1 + \lambda_2 = 2\lambda_1. \] Then, since \( x \geq 0 \) and \( \lambda_2 \geq 0 \) by (11) and (12), we get \( 2\lambda_1 \geq 1 > 0 \), which implies, by (8), \( 2x + 2y - 1 = 0 \). Look at the case where \( \lambda_2 > 0 \). Thus, by (9) \( x = 0 \). Then since \( 2x + 2y = 1 \) we get \( y = 1/2 \). By (10), \( \lambda_3 = 0 \). Plug \( y = 1/2 \) and \( \lambda_3 = 0 \) into (7) to get \( \lambda_1 = 2 \). Then plug \( x = 0 \) and \( \lambda_1 = 2 \) into (6) to get \( \lambda_2 = 3 \). Thus one KTK point is \( (x, y, \lambda_1, \lambda_2, \lambda_3) = (0, 1/2, 2, 3, 0) \). Now look at the case where \( \lambda_2 = 0 \). (6) and (7) together imply \( 2x + 1 + \lambda_2 = 8y + \lambda_3 = 2\lambda_1 \), thus \( 2x + 1 = 8y + \lambda_3 \). Substitute \( 2x = 1 - 2y \) and get \( 1 - 2y + 1 = 8y + \lambda_3 \), that is, \( 2 = 10y + \lambda_3 \). Then, \( \lambda_3y = 0 \) implies either \([y = 1/5 \text{ and } \lambda_3 = 0]\) or \([y = 0 \text{ and } \lambda_3 = 2]\). In the former case, we get \( x = 3/10 \) and \( \lambda_1 = 4/5 \). In the latter case, we get \( x = 1/2 \) and \( \lambda_1 = 1 \). Thus we get two more KTK points: \( (x, y, \lambda_1, \lambda_2, \lambda_3) = (3/10, 1/5, 4/5, 0, 0) \) and \( (x, y, \lambda_1, \lambda_2, \lambda_3) = (1/2, 0, 1, 0, 2) \).