In this lecture we first discuss metric spaces and open and closed sets. We give characterizations of open and closed sets through neighborhoods and through sequences. Then, we discuss continuity of functions, together with uniform continuity. We also discuss connected metric spaces, separable metric spaces.

1 Generalizing the distance concept: Metric Space

**Definition 1** $X \neq \emptyset$. A function $d : X \times X \to \mathbb{R}_+$ is called a metric (or a distance function) if for all $x, y, z \in X$,

(i) $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$ (symmetry)

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

$(X, d)$ is called a metric space when $d$ is a metric on $X$.

**Example 1** $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$.

**Example 2** $d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ is called the "discrete metric". Check!!

**Example 3** Let $X = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ and let $d(x, y)$ be the length of the shorter arc in $X$ that join $x$ an $y$. Check!!
Example 4 Consider $\mathbb{R}^2$. One way to put a metric on it is through $d_2(x, y) = (\sum_{i=1}^{2} |x_i - y_i|^2)^{\frac{1}{2}} = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{\frac{1}{2}}$

More generally $(\mathbb{R}^n, d_p)$ is a metric space $\forall 1 \leq p < \infty$ where $d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ is defined by $d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}$

Consider the unit circle in $\mathbb{R}^2$; $(n = 2)$.

$$C_p = \{x \in \mathbb{R}^2 : d_p((0, 0), x) = 1\}$$

That is,

$$C_1 = \{x \in \mathbb{R}^2 : |x_1 - y_1| + |x_2 - y_2| = 1\} = \{x \in \mathbb{R}^2 : |x_1| + |x_2| = 1\}$$

$$C_2 = \{x \in \mathbb{R}^2 : (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{\frac{1}{2}} = 1\} = \{x \in \mathbb{R}^2 : |x_1|^2 + |x_2|^2 = 1\}$$

Suppose for $p = \infty$ we define

$$d_\infty(x, y) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$$

Then, $C_\infty = \{x \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} = 1\}$

That is,
Also note that;

$(\mathbb{R}^n,d_2)$ is called the $n$-dimensional Euclidean space.

**Example 5** Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-1 function.

Define $d(x,y) = |f(x) - f(y)| \forall x,y \in \mathbb{R}.$

Then, $d$ is a metric on $\mathbb{R}$. Check!!

## 2 Open and Closed Sets

**Definition 2** Let $(X,d)$ be a metric space.

For any $x \in X$ and $\epsilon > 0$, $\epsilon$-neighborhood of $x$ in $X$ is the set $N_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$.

A neighborhood of $x$ in $X$ is any subset of $X$ that contains at least one $\epsilon$-neighborhood of $x$ in $X$.

Note that $N_\epsilon(x)$ depends not only $\epsilon$ and $x$ but also on $X$ and $d$. For instance, $N_1(0)$ in $\mathbb{R}^2$ is $\{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ but $N_1(0)$ in $\mathbb{R} \times \{0\}$ is $\{(x_1,0) \in \mathbb{R}^2 : -1 < x_1 < 1\}$. Similarly, $N_1(0)$ in $\mathbb{R}^2$ is different from that in $\mathbb{R}^{2,p}$ for $p \neq 2$.

**Definition 3** A subset $S \subseteq X$ is open in $X$ if for all $x \in S$, there exists an $\epsilon > 0$ such that $N_\epsilon(x) \subseteq S$. A subset $S \subseteq X$ is closed in $X$ if $X \setminus S$ is open.
Example 6 \( \{ x : -1 < x < 1 \} \subseteq \mathbb{R} \) is open in \( \mathbb{R} \).

\( \{ x : -1 < x < 1 \} \subseteq \mathbb{R}^2 \) is not open in \( \mathbb{R}^2 \) since \( N_\varepsilon(0,0) \notin (-1,1) \forall \varepsilon > 0 \) and not closed in \( \mathbb{R}^2 \) since complement is not open.

Example 7 \( S = \{ n + \frac{1}{n} : n = 1, 2, 3, \ldots \} \subseteq \mathbb{R} \) is closed because \( \mathbb{R} \setminus S \) is open.

Example 8 \( S = \{ (\frac{1}{n}, \frac{n+1}{n}) : n = 1, 2, 3, \ldots \} \subseteq \mathbb{R}^2 \) is neither open nor closed.

- \( x = (1,2) \ N_\varepsilon(x) \not\subseteq S \ \forall \varepsilon > 0 \Rightarrow \) not open.
- \( \mathbb{R}^2 \setminus S \) is not open since \( N_\varepsilon(0,1) \not\subseteq \mathbb{R}^2 \setminus S \ \forall \varepsilon > 0 \Rightarrow \) not closed.

Proposition 1.

(i) \( \bigcap_{i=1}^{\infty} S_i \) open when all \( S_i \) are open. (That’s how we know \( \text{Int}(S) \) is well defined.)

(ii) \( \bigcap_{i=1}^{\infty} S_i \) closed when all \( S_i \) are closed. (That’s how we know \( \text{Cl}(S) \) is well defined.)

(iii) \( \bigcap_{i=1}^{n} S_i \) open when all \( S_i \) are open.

(iv) \( \bigcup_{i=1}^{n} S_i \) closed when all \( S_i \) are closed.

(v) \( \bigcap_{i=1}^{\infty} S_i \) need not be open even if \( S_i \)’s are all open.

(vi) \( \bigcup_{i=1}^{\infty} S_i \) need not be closed even if \( S_i \)’s are all closed.

(vii) \( S \) is open in \( (X,d) \Leftrightarrow S \) is open in \( (X, \frac{d}{1+d}) \).

Proof of (iii). Let \( S = \bigcap_{i=1}^{n} S_i \) with \( S_i \) open \( \forall i = 1, \ldots, n \). Let \( x \in S \). Then \( x \in S_i \forall i = 1, \ldots, n \). Since \( S_i \) is open \( \exists N_\varepsilon(x) \subseteq S_i \). Define \( \epsilon = \min \{ \epsilon_i \}_{i=1}^{n} \). Then \( N_\epsilon(x) \subseteq S_i \forall i = 1, \ldots, n \). So, \( N_\epsilon(x) \subseteq \bigcap_{i=1}^{n} S_i = S \). ■

Proof of (iv). Let \( S = \bigcup_{i=1}^{n} S_i \) with \( S_i \) closed. Then, \( X \setminus S = \bigcap_{i=1}^{n} (X \setminus S_i) \). Since \( S_i \) closed, we have \( X \setminus S_i \) open. Then \( X \setminus S \) is open by (iii) above. Hence \( S \) is closed. ■

Counterexamples for (v).

Example 9 \( \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 + \frac{1}{n} \right) \) \( \begin{array}{c} \text{open} \end{array} \) \( \begin{array}{c} \text{closed} \end{array} \)

Example 10 \( \bigcap_{n=1}^{\infty} (-1, \frac{1}{n}) \)

Counterexamples for (vi).

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Example 11 \( \bigcup_{n=1}^{\infty} \left[ -\frac{1}{n}, \frac{1}{n} \right] = \mathbb{R} \) is open.

Example 12 \( \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = (-\infty, 1] \) is not closed.

Example 13 \( \bigcup_{n=1}^{\infty} \left[ 0, 1 - \frac{1}{n} \right] = [0, 1) \)

Example 14 Given \((X, d)\); \(X\) and \(\emptyset\) are clopen. Why?

Example 15 Given \((X, d)\); for any \(x \in X\) and \(\epsilon > 0\), \(N_\epsilon(x)\) is open and \(\{x\}\) is closed. Why?

Example 16 Let \( \emptyset \neq S \subseteq \mathbb{R} \) be open in \(\mathbb{R}\). Let \( \emptyset \neq A \subseteq \mathbb{R} \) be any subset in \(\mathbb{R}\). Then \(A + S\) is open where \(A + S = \{x = a + s : a \in A, s \in S\}\). \textbf{Proof.} Take any \(a + s \in A + S\). There exists an \(\epsilon > 0\) s.t. \(N_\epsilon(s) = (s - \epsilon, s + \epsilon) \subseteq S\). Now take any \(x \in N_\epsilon(a + s)\). Then, \(x\) is of the form \(x = a + s'\) where \(s' \in (s - \epsilon, s + \epsilon) \subseteq S\). So \(x \in A + S\). Thus \(N_\epsilon(a + s) \subseteq A + S\). \(\blacksquare\)

Example 17 Let \( \emptyset \neq A, B \subseteq \mathbb{R} \) be both closed in \(\mathbb{R}\). Then \(A + B\) is not necessarily closed. \textbf{Proof.} To see this we only need a counterexample. \(A = \{n + \frac{1}{n} : n = 2, 3, \ldots\}\) is closed. \(B = \{-m : m = 2, 3, \ldots\}\) is closed. \(A + B = \{n - m + \frac{1}{n} : n, m = 2, 3, \ldots\}\). \(\mathbb{R} \setminus A + B\) is not open since \(\exists \epsilon > 0\) s.t. \(N_\epsilon(0) \subseteq \mathbb{R} \setminus A + B\). \(\blacksquare\)

Example 18 Let \((X, d)\) be a metric space where \(d\) is the discrete metric. Then any \(S \subseteq X\) is open. Why? Pick \(x \in S \subseteq X\). Then \(N_{\frac{1}{2}}(x) = \{x\} \subseteq S\). Note, \(X \setminus S\) is open for any \(S\), too. Thus any \(S \subseteq X\) is also closed.

\textbf{Definition 4} Let \((X, d)\) be a metric space, and \(S \subseteq X\).

(i) The largest open set contained in \(S\) is called the \textbf{interior} of \(S\), denoted by \(\text{Int}(S)\), that is, \(\text{Int}(S) = \bigcup_T \{T : T \subseteq S, T \text{ is open}\}\), that is if \(T \subseteq S\) and \(T\) is open then \(T \subseteq \text{Int}(S)\).

(ii) The smallest closed set containing \(S\) is called the \textbf{closure} of \(S\), denoted by \(\text{Cl}(S)\), that is \(\text{Cl}(S) = \bigcap_T \{T : S \subseteq T, T \text{ is closed}\}\), that is if \(S \subseteq T\) and \(T\) is closed then \(\text{Cl}(S) \subseteq T\).

(iii) The \textbf{boundary} of \(S\), denoted by \(\text{Bd}(S)\), is defined as \(\text{Bd}(S) = \text{Cl}(S) \setminus \text{Int}(S)\).

\textbf{Remark 1} \(\text{Int}(S) \subseteq S\) is open for any \(S\). \(\text{Cl}(S) \supseteq S\) is closed for any \(S\).
Remark 2 Define the exterior of \( S \) to be \( \text{Ext}(S) = \text{Int}(X\setminus S) \). Then \( \text{Bd}(S) = X\setminus(\text{Int}(S) \cup \text{Ext}(S)) \).

Remark 3 \( \text{Cl}(S) = S \cup \text{Bd}(S) \)

Example 19 \( S = (0, 1] \subseteq \mathbb{R} \). Then \( \text{Int}(S) = (0, 1) \) and \( \text{Cl}(S) = [0, 1] \) and \( \text{Bd}(S) = \{0, 1\} \).

Example 20 \( S = \{(x, y) : x^2 + y^2 < 1\} \cup \{(1, 0)\} \subseteq \mathbb{R}^2 \).
  \( \text{Int}(S) = \{(x, y) : x^2 + y^2 < 1\} \)
  \( \text{Cl}(S) = \{(x, y) : x^2 + y^2 \leq 1\} \)
  \( \text{Bd}(S) = \{(x, y) : x^2 + y^2 = 1\} \)

Example 21 Given \((X, d)\) a metric space, \( S \subseteq X \) is
  (i) closed iff \( S = \text{Cl}(S) \)
  (ii) open iff \( S = \text{Int}(S) \)

Proof. (i) \( \Rightarrow \) If \( S \) is closed, then it is already the smallest closed set containing \( S \). So \( S = \text{Cl}(S) \).

\( \Leftarrow \) If \( S = \text{Cl}(S) \), then by definition of \( \text{Cl}(S) \), \( S \) is closed.

(ii) \( \Rightarrow \) If \( S \) is open, then it is already the largest open set contained in \( S \). So \( S = \text{Int}(S) \).

\( \Leftarrow \) If \( S = \text{Int}(S) \), then by definition of \( \text{Int}(S) \), \( S \) is open. ■

Theorem 1 (Characterizations using Neighborhoods/Balls) Let \((X, d)\) be a metric space.

Then,

(i) \( x \) is an accumulation point of \( S \subseteq X \) \( \iff \) \( \forall \epsilon > 0 \ (N_\epsilon(x)\setminus \{x\}) \cap S \neq \emptyset \).

(ii) \( S \subseteq X \) is open \( \iff \forall x \in S \ \exists \epsilon > 0 \ \text{s.t.} \ N_\epsilon(x) \subseteq S \)

(iii) \( X \in \text{Int}(S) \) \( \iff \exists \epsilon > 0 \ \text{s.t.} \ N_\epsilon(x) \subseteq S \)

(iv) \( X \in \text{Cl}(S) \) \( \iff \forall \epsilon > 0 \ N_\epsilon(x) \cap S \neq \emptyset \)

(v) \( X \in \text{Bd}(S) \) \( \iff \forall \epsilon > 0 \ N_\epsilon(x) \cap S \neq \emptyset \) and \( N_\epsilon(x) \cap (X\setminus S) \neq \emptyset \)

(vi) \( X \in \text{Ext}(S) \) \( \iff \exists \epsilon > 0 \ \text{s.t.} \ N_\epsilon(x) \subseteq (X\setminus S) \)

Proof. Exercise. Most of them are trivially implied by definitions! ■

3 Sequences in a Metric Space

Definition 5 Let \((X, d)\) be a metric space. Let \((x_n)\) be a sequence in \( X \). Then, \((x_n)\) converges to \( x \) if, \( \forall \epsilon > 0 \ \exists N \in \mathbb{N} \) such that \( d(x_n, x) < \epsilon \ \forall n \geq N \).
Remark 4 If \( x_n \to x \) and \( x_n \to y \), then \( x = y \). Why?

Proposition 2 A subset \( S \subseteq X \) is closed if and only if every sequence in \( S \) that converges in \( X \) converges to a point in \( S \).

Proof. \( \Rightarrow \) Let \( S \) be a closed subset of \( X \). Take any \( (x_n) \) in \( S \) with \( x_n \to x \) for some \( x \in X \). Suppose \( x \in X \setminus S \). Then we can find an \( \epsilon > 0 \) with \( N_\epsilon(x) \subseteq X \setminus S \) since \( X \setminus S \) is open. But since \( d(x_n, x) < \epsilon \) for all \( n \geq N \) for some \( N \in \mathbb{N} \), we get \( x_N, x_{N+1}, \ldots \in X \setminus S \) contradicting \( x_n \) is in \( S \) for all \( n \).

\( \Leftarrow \) Suppose that \( S \) is not closed in \( X \). Then \( X \setminus S \) is not open. So \( \exists x \in X \setminus S \) such that every \( N_\epsilon(x) \) intersects \( S \). Thus, for any \( n = 1, 2, \ldots \), there exist an \( x_n \in N_\frac{1}{n}(x) \cap S \). But then \( x_n \) is a sequence in \( S \) with \( x_n \to x \), and yet \( x \notin S \).

Thus if \( S \) was not closed, there would exist at least one sequence in \( S \) that converges to a point outside of \( S \). \( \blacksquare \)

Theorem 2 (Characterizations using Sequences) Let \( (X, d) \) be a metric space. Then,

(1) \( x \) is an accumulation point of \( S \subseteq X \) \( \iff \exists (x_n) \) with distinct terms \( x_n \in S \) such that \( \lim x_n = x \).

(2) \( S \subseteq X \) is closed \( \iff \forall (x_n) \) in \( S \) with \( \lim x_n = x \), we have \( x \in S \).

(3) \( x \in Cl(S) \) \( \iff \exists (x_n) \) in \( S \) such that \( \lim x_n = x \).

(4) \( x \in Bd(S) \) \( \iff \exists (x_n, (x'_n)) \) where \( (x_n) \) is in \( S \) and \( (x'_n) \) is in \( X \setminus S \) such that \( \lim x_n = \lim x'_n \).

Proof. (1) \( \Leftarrow \) By definitions of acc. pt. and limit.

\( \Rightarrow \) If \( x \) is an acc. pt., then \( \forall \epsilon > 0 \ (N_\epsilon(x) \setminus \{x\}) \cap S \neq \emptyset \). Let \( x_n \in (N_\frac{1}{n}(x) \setminus \{x\}) \cap S \). Check \( \lim x_n = x \).

(2) We’ve just proved this in above proposition.

(3) \( \Rightarrow \) If \( x \in Cl(S) \), then for any \( n \in \mathbb{N} \ N_\frac{1}{n}(x) \cap S \neq \emptyset \). If \( x_n \) is chosen from this set, we get \( \lim x_n = x \) since \( d(x_n, x) < \frac{1}{n} \).

\( \Leftarrow \) If \( \lim x_n = x \) where \( x_n \) is in \( S \), then \( \forall \epsilon > 0 \) we find \( N \) with \( d(x_n, x) < \epsilon \ \forall n \geq N \). So \( N_\epsilon(x) \cap S \neq \emptyset \) since \( x_n \) belongs to this intersection \( \forall n \geq N \). Thus \( x \in Cl(S) \).

(4) Exercise. \( \blacksquare \)
4 Continuity of Functions

Idea 1: Images of points nearby \( x \) under \( f \) are close to \( f(x) \).

Idea 2: Small perturbation in the input entails onlay a small perturbation in the output.

**Definition 6** Let \((X, d_x)\) and \((Y, d_y)\) be two metric spaces.

The function \( f : X \to Y \) is **continuous at** \( x \in X \)
if
\[
\begin{align*}
\text{if } & \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \\
d_x(x, y) < \delta & \text{ implies } d_y(f(x), f(y)) < \epsilon \text{ for each } y \in X.
\end{align*}
\]

i.e.
\[
\begin{align*}
\text{if } & \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \\
d_y(f(x), f(y)) < \epsilon & \text{ for all } y \in N_{\delta,X}(x)
\end{align*}
\]

i.e.
\[
\begin{align*}
\text{if } & \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \\
f(N_{\delta,x}(x)) & \subseteq N_{\epsilon,Y}(f(x))
\end{align*}
\]

The function \( f : X \to Y \) is **continuous on** \( X \) if it is continuous at every \( x \in X \).

**Remark 5** Note that \( \delta \) may depend on both \( \epsilon \) and \( x \).

![Diagram showing continuity of a function](image)

**Example 22** Let \( f : [0, 2] \to \mathbb{R} \) with \( f(x) = x^2 \).

Given \( \epsilon > 0 \) choose \( \delta = \epsilon/5 \).
If $d(x, y) = |x - y| < \delta = \epsilon/5$, then

$$d(f(x), f(y)) = |f(x) - f(y)| = |x^2 - y^2| \leq |x - y||x + y| \leq 4|x - y| < 4\epsilon/5 < \epsilon$$

**Example 23** Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.

*Given $\epsilon > 0$ choose $\delta$ s.t. $\delta^2 + 2\delta|x| < \epsilon$ i.e. $\delta^2 + 2\delta|x| - \epsilon < 0$ If $d(x, y) = |x - y| < \delta$, then $|y| - |x| \leq |y - x| = |x - y| < \delta$. So $|y| < \delta + |x|$. Then $|x^2 - y^2| \leq |x - y||x + y| < \delta(|x| + |y|)$. So $|x^2 - y^2| < \delta(2|x| + \delta) = \delta^2 + 2\delta|x| < \epsilon$ for $\delta$ selected above.*

**Example 24** Let $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $f(x) = 1/x$.

*First observe that $|f(x) - f(y)| = |1/x - 1/y| = |x - y|/xy$. Given $\epsilon > 0$, we want to find a $\delta > 0$ such that $|x - y|/xy < \epsilon$ for any $y > 0$ when $|x - y| < \delta$. Note that $|x - y| < \delta/xy$ $\forall 0 < \delta < x$ with $|x - y| < \delta$. But $\delta/(x(x - \delta)) < \epsilon$ if $\delta < \epsilon x^2/(1 + x^2)$. So by choosing any $\delta > 0$ with $\delta < \epsilon x^2/(1 + x^2)$, we find $d(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. Notice that $\delta = \delta(\epsilon, x)$.*

### 4.1 Uniform Continuity

Continuity with $\delta$ not depending on $x$, that is $\delta = \delta(\epsilon)$, rather than $\delta = \delta(\epsilon, x)$.

**Definition 7** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. The function $f : X \to Y$ is uniformly continuous if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $f(N_{\delta,X}(x)) \subseteq N_{\epsilon,Y}(f(x)) \forall x \in X$.

**Remark 6** $\delta$ is independent of $x \in X$; $\delta = \delta(\epsilon)$.

**Remark 7** If $f$ is uniformly continuous, then it is obviously continuous.

**Example 25** $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is continuous, but it is not uniformly continuous. To see the latter, take $\epsilon = 1$, and for any $\delta > 0$ let $x_n = n$ and $x'_n = n + \frac{\delta}{2}$. Then $|x_n - x'_n| < \delta$, but $|f(x_n) - f(x'_n)| = |x_n + x'_n||x_n - x'_n| = 2n + \frac{\delta}{2} > n\frac{\delta}{2} = n\delta$. This means that, $\forall \delta > 0$ we can find $x, y$ with $|x - y| < \delta$ but $|f(x) - f(y)| > 1$. (Note that $f(x) = x^2$ is uniformly continuous if $f : [0, 2] \to \mathbb{R}$.)
Example 26 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = \sin x$ is uniformly continuous. Given $\epsilon > 0$, choosing $\delta = \epsilon$ we have; if $|x - y| < \delta = \epsilon$ then $|\sin x - \sin y| \leq |x - y| < \delta = \epsilon$, where the weak inequality follows from the MVT: $|\sin x - \sin y| = |\cos(z)(x - y)| \leq |x - y|$

Note: The intuition is as follows. For instance $f(x) = 1/x$ is continuous but not uniformly continuous. It is harder to find $\delta$ nearby 0. If it were uniformly continuous, one $\delta(\epsilon)$ would always work.

4.2 A fundamental Characterization of Continuity

Proposition 3 Let $(X, d_x)$ and $(Y, d_y)$ be two metric spaces. Let $f : X \rightarrow Y$.

Then, the followings are equivalent;

(i) $f$ is continuous.  
(ii) $\forall S \text{ open in } Y, f^{-1}(S)$ is open in $X$.  
(iii) $\forall S \text{ closed in } Y, f^{-1}(S)$ is closed in $X$.  
(iv) $\forall x \in X, (x_n)$ in $X$, if $\lim x_n = x$ then $\lim f(x_n) = f(x)$.  

Proof. (i) $\Rightarrow$ (ii): Let $S \subseteq Y$ be open in $Y$. Take $x \in f^{-1}(S)$. Then $f(x) \in S$. Since $S$ is open, $\exists \epsilon > 0$ such that $N_{\epsilon,Y}(f(x)) \subseteq S$. By continuity of $f$ at $x$, $\exists \delta > 0$ such that $f(N_{\delta,X}(x)) \subseteq N_{\epsilon,Y}(f(x)) \subseteq S$. Thus $N_{\delta,X}(x) \subseteq f^{-1}(S)$. Hence $f^{-1}(S)$ is open in $X$.

(ii) $\Rightarrow$ (iv): Take any $x \in X$, any $(x_n)$ in $X$ with $\lim x_n = x$. Fix an arbitrary $\epsilon > 0$. Note that $N_{\epsilon,Y}(f(x))$ is open. Then by (ii) $f^{-1}(N_{\epsilon,Y}(f(x)))$ is also open in $X$. Note that $x \in f^{-1}(N_{\epsilon,Y}(f(x)))$. Now, $\lim x_n = x$ implies that $\exists N$ s.t. $x_n \in f^{-1}(N_{\epsilon,Y}(f(x))) \ \forall n \geq N$. Thus $f(x_n) \in N_{\epsilon,Y}(f(x)) \ \forall n \geq N$.

(iv) $\Rightarrow$ (i): Assume that $f$ is not continuous. We’ll show that for some sequence $(x_n)$ with $\lim x_n = x$, $\lim f(x_n) \neq f(x)$. Now, $f$ not continuous means that $\exists \epsilon > 0$ such that $\forall \delta > 0$, we can find a point $x' \in X$ with $d_x(x, x') < \delta$, but $d_y(f(x), f(x')) \geq \epsilon$. So for $\delta = 1/n \ \exists x_n$ with $d_x(x, x_n) < 1/n$ but $d_y(f(x), f(x_n)) \geq \epsilon \ \forall n \in \mathbb{N}$. Thus $\lim x_n = x$, but the sequence $f(x_n)$ can not have $f(x)$ as its limit.

So far we have shown (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv). To complete the proof we need to show that (iii) is equivalent to one of the (i), (ii) and (iv). Let’s show (ii) $\Leftrightarrow$ (iii).

(ii) $\Leftrightarrow$ (iii): If $S$ is closed in $Y$, then $Y \setminus S$ is open in $Y$. So (ii) implies $f^{-1}(Y \setminus S)$ is open in $X$. 

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Since $X \setminus f^{-1}(S) = f^{-1}(Y \setminus S)$, this means that $f^{-1}(S)$ is closed. Necessity is shown analogously.

5 Connectedness and Separability

5.1 Connectedness

Intuitively, a connected subset of a metric space is one that cannot be partitioned into separate pieces; rather than it is one whole piece.

In $\mathbb{R}$, for instance, $(0, 1)$ is connected but $[0, 1] \cup [2, 3)$ is not.

**Definition 8** A metric space $(X, d)$ is connected if there is no two nonempty and disjoint open subsets $S$ and $S'$ of $X$ such that $S \cup S' = X$.

A subset $S$ of $X$ is connected in $X$ if $\forall S', S'' \subseteq X$ such that $S'$ and $S''$ are open and $S \subseteq S' \cup S''$ with $S \cap S' \cap S'' = \emptyset$ and $S \cap S' \neq \emptyset$ and $S \cap S'' \neq \emptyset$.

**Proposition 4** Let $(X, d)$ be a metric space. Then, $X$ is connected if and only if the only clopen subsets of $X$ are $\emptyset$ and $X$.

**Proof.** $\Rightarrow$ If $S \notin \{\emptyset, X\}$ is a clopen subset of $X$, then $X$ cannot be connected since $X = S \cup (X \setminus S)$.

$\Leftarrow$ Conversely, assume that $X$ is not connected. In this case there are two nonempty disjoint open subsets $S$ and $S'$ of $X$ such that $S \cup S' = X$. But then $S' = X \setminus S$ so that $S$ must be both open and closed. Since $S \notin \{\emptyset, X\}$, this proves the result.

**Example 27** Consider $(\mathbb{R}^n, d_2)$ the Euclidean metric space

1) $\mathbb{R}^n$ is connected because the only clopen sets are $\emptyset$ and $\mathbb{R}^n$.

2) Any interval in $\mathbb{R}$ is connected.

**Proof.** Let $I \subseteq \mathbb{R}$. Suppose $I$ is not an interval. Then $\exists s_1, s_2 \in I$ and $\exists c \in \mathbb{R}$ such that $s_1 < c < s_2$ and $c \notin I$. Let $S' = (-\infty, c)$, $S'' = (c, \infty)$. Note $S'$ and $S''$ are open. Also $I \subseteq \mathbb{R} \setminus c = S' \cup S''$. Moreover $I \cap S' \cap S'' = \emptyset$ and $s_1 \in I \cap S' \neq \emptyset$ and $s_2 \in I \cap S'' \neq \emptyset$. Thus $I$ is not connected.
Proposition 5 Let \((X, d)\) be a metric space. Let \(A \subseteq X\) be a connected subset. Then any \(E \subseteq X\) with \(A \subseteq E \subseteq Cl(A)\) is connected. In particular, \(Cl(A)\) is connected.

Proof. Let \(A \subseteq E \subseteq Cl(A)\) and suppose \(E\) is not connected. So \(\exists\) open sets \(S'\) and \(S''\) such that \(E \subseteq S' \cup S''\), \(E \cap S' \neq \emptyset\), \(E \cap S'' \neq \emptyset\) and \(E \cap S' \cap S'' = \emptyset\). So \(A \subseteq S' \cup S''\) since \(A \subseteq E \subseteq S' \cup S''\). Also \(A \cap S' \cap S'' = \emptyset\) since \(A \cap S' \cap S'' \subseteq E \cap S' \cap S'' = \emptyset\). Now if both \(A \cap S' \neq \emptyset\) and \(A \cap S'' \neq \emptyset\) hold then \(A\) would not be connected. Thus one of them must be empty. Say \(A \cap S' = \emptyset\). But \(E \cap S' \neq \emptyset\). So \(\exists x \in (E \setminus A) \cap S'\) and \(x \in E \subseteq Cl(A)\). Thus \(x \in Bd(A)\). Since \(x \in S'\) open, \(\exists N(x) \subseteq S'\). Since \(S' \subseteq X \setminus A\) we get \(N(x) \subseteq X \setminus A\) contradicting \(x \in Bd(A)\). □

Proposition 6 Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. Let \(f : X \rightarrow Y\) be a continuous function. Then, if \(X\) is connected, then \(f(X)\) is a connected subset of \(Y\).

Proof. If \(f(X)\) is not connected in \(Y\), then \(\exists\) open sets \(S, S'\) s.t. \(f(X) \subseteq S \cup S', f(X) \cap S \cap S' = \emptyset\), \(S \cap f(X) \neq \emptyset\), \(S' \cap f(X) \neq \emptyset\). Since \(f\) is continuous and \(S, S'\) are open, we have \(f^{-1}(S)\) and \(f^{-1}(S')\) are open in \(X\). Also we have \(X \subseteq f^{-1}(S \cup S') = f^{-1}(S) \cup f^{-1}(S')\). Also \(X \cap f^{-1}(S) \cap f^{-1}(S') = \emptyset\). Otherwise \(\exists x \in X \cap f^{-1}(S) \cap f^{-1}(S')\), that is \(f(x) \in f(X) \cap S \cap S'\). Also \(f^{-1}(S) \cap X \neq \emptyset\) and \(f^{-1}(S') \cap X \neq \emptyset\). Contradiction! □

Theorem 3 (Intermediate Value theorem) Let \((X, d)\) be a connected metric space. Let \(f : X \rightarrow \mathbb{R}\) be a continuous function. If \(f(x) \leq a \leq f(y)\) for some \(x, y \in X\), then \(\exists z \in X\) s.t. \(f(z) = a\).

Proof. By above proposition \(f(X)\) is connected in \(\mathbb{R}\). Thus \(f(X)\) must be an interval. Then there exists \(z \in X\) s.t. \(f(z) = a\) where \(f(x) \leq a \leq f(y)\) for some \(x, y \in X\). □

5.2 Separability

Definition 9 Let \((X, d)\) be a metric space and \(Y \subseteq X\). If \(Cl(Y) = X\) then \(Y\) is dense in \(X\). \(X\) is separable if it contains a countable dense set.
Remark 8 X separable iff $\exists$ countable $Y \subseteq X$ such that $x \in X \iff \exists (y_n)$ in $Y$ with $\lim y_n = x$.

Example 28 $\mathbb{R}$ is separable since $\text{Cl}(\mathbb{Q}) = \mathbb{R}$ and $\mathbb{Q}$ is countable.

Proposition 7 Let $(X,d)$ be a metric space.

A continuous function $f : X \to Y$ is determined by its values on a dense subset of $X$.

Proof. That is; if $\text{Cl}(A) = X$ and $f, g$ are continuous functions from $X$ into $Y$, then $f(x) = g(x) \ \forall x \in A$ implies $f = g$. For each $x \in X = \text{Cl}(A)$, there is a sequence $x_n$ in $A$ such that $\lim x_n = x$. But then by the fundamental characterization of continuity; we get $f(x) = \lim f(x_n) = \lim g(x_n) = g(x)$. ■