Do not forget to write your full name, student number and section on the top.

Turn off your cell phone and put it away. During the exam if you are seen with a cell phone, on or off, 20 points will be taken off from your exam immediately.

Put away all your lecture notes, books, etc.

There are 5 questions and 10 pages in the exam. Make sure you have them all.

Please answer all of the questions in the space provided for each question.

Show your work!

You have 150 minutes.

GOOD LUCK!!
1. (18 pts) Behzat Ç., the head of the murder investigation unit in Ankara Police Department, wants to buy a red 1974 made Volkswagen Beetle for her daughter as a birthday present. He goes to the second hand car market and finds a nice looking, handsome red Beetle. Behzat agrees with the seller on the following terms: They will both name one of the two possible prices, 1000TL or 4000TL, simultaneously. If they name the same price, then trade occurs at that price. If they name different prices and Behzat names the higher price, then trade occurs at the average price, 2500TL. If they name different prices and the seller names the higher price, then trade does not occur. If trade occurs at price \( P \), then Behzat’s payoff is \( V - P \) and the seller’s payoff is \( P \). If trade does not occur, then Behzat’s payoff is 0 and the seller’s payoff is \( R \). For Behzat, \( V \) can take two values, it’s either 600 with probability \( \frac{1}{3} \), or 4200 with probability \( \frac{2}{3} \). For the seller, \( R \) can take two values, it’s either 1200 or 1500 with equal probabilities. Behzat learns his \( V \) value and the seller learns his \( R \) value but they do not observe the other’s value.

(a) (4 pts) Write down the strategy sets for both Behzat and the seller.

**ANSWER:**

\[ S_B = S_S = \{(1000, 1000), (1000, 4000), (4000, 1000), (4000, 4000)\} \]

For instance, for Behzat (1000,4000) means 1000 if \( V \) is 600 and 4000 if \( V \) is 4200, and for the seller, (1000,4000) means 1000 if \( R \) is 1200 and 4000 if \( R \) is 1500.

(b) (14 pts) Find the set of pure strategy Bayesian Nash equilibrium.

**ANSWER:**

The game is as follows

\[
\begin{array}{c|cc}
\text{Seller/Behzat} & 1000 & 4000 \\
\hline
1000 & 1000, V-1000 & 2500, V-2500 \\
4000 & R,0 & 4000, V-4000 \\
\end{array}
\]

For the seller, since \( R > 1000 \) for both of its values, 4000 dominates 1000 for each type. Thus we can eliminate all of the seller’s strategies except (4000,4000).

For Behzat, if \( V = 600 \) then 1000 dominates 4000, and if \( V = 4200 \) then there is no domination. Thus we can eliminate (4000,1000) and (4000,4000) for Behzat.

Given (4000,4000) for the seller, the expected payoff for Behzat from (1000,1000) is 0, and the expected payoff for Behzat from (1000,4000) is \((1/3) \cdot 0 + (2/3) \cdot (4200 - 4000) = 400/3\). Thus, for Behzat it’s optimal to play (1000,4000).

Thus, the unique pure strategy Bayesian Nash equilibrium is \((4000,4000),(1000,4000)\) where the first strategy is of the seller and the second one is of Behzat.
2. (20 pts) Harun and Cevdet, who are in Behzat’s investigation team, are bargaining over a cake of size 12. In period 1, Harun offers $m_1 \in [0, 12]$ to Cevdet. Cevdet accepts or rejects. If it is rejected, Cevdet offers $m_2 \in [0, 12]$ to Harun in period 2. Harun accepts or rejects. If this offer is also rejected, Harun makes an offer $m_3 \in [0, 12]$ to Cevdet in period 3, who then accepts or rejects it. If the period 3 offer is rejected the cake is given to Eda (who is not a player here), and Harun receives zero payoff while Cevdet receives a payoff of -1 (since he gets beaten up by Harun). If an offer $m_t$ is accepted in period $t$, the player making the offer receives payoff $(3/4)^{t-1}(12 - m_t)$, and the receiver of the offer gets $(3/4)^{t-1}m_t$, where $3/4$ is their common discount factor.

(a) (4 pts) Describe the strategy set for Cevdet.

ANSWER:
Cevdet has to specify whether to accept or reject any possible offer $m_1 \in [0, 12]$, denote this by $f : [0, 12] \rightarrow \{A, R\}$. He also needs to specify an offer $m_2$ in case of rejecting offer $m_1$, denote this by $g : [0, 12] \rightarrow [0, 12]$. Finally, he has to specify whether to accept or reject any possible offer $m_3 \in [0, 12]$, denote this by $h : [0, 12] \rightarrow \{A, R\}$. Thus, a strategy for Cevdet consists of three functions: $s_{cev} = (f, g, h)$.

(b) (16 pts) Find the unique pure strategy Subgame Perfect Nash equilibrium. What will be Cevdet’s payoff in the equilibrium?

ANSWER:
There is a unique SPNE. In period 3, Harun offers $m_3 = 0$ and Cevdet accepts this. So in period 2, Harun accepts any offer $m_2 \geq \frac{3}{4} \cdot 12 = 9$. Thus, Cevdet offers $m_2 = 9$ to Harun, keeping 3 for himself. Thus, Cevdet in period 1 accepts any offer $m_1 \geq \frac{3}{4} \cdot 3$. This means Harun offers $m_1 = 9/4$ and Cevdet accepts. Thus, Cevdet will end up with $9/4$. 

3
3. (22 pts) Behzat Ç. is investigating a murder crime scene with his team. The only evidence they find is a diagram given below. Little they know, it’s an extensive form game with three players.

(a) (12 pts) Find the set of pure strategy Perfect Bayesian equilibrium.

**ANSWER:**

Fix $u$ for player 3. Then, for player 1 $r$ is optimal. Then for player 2 $c$ dominates $d$ ($3 > 1$ and $5 > 2$). Then for player 1 $e$ is optimal. Thus, we obtain $(er, c, u)$. With this strategy profile $r = 1$ by requirement 4. But then at this info set with $r = 1$ optimal action is $n$. Thus player 3 is not sequentially rational. No such PBE!

Fix $n$ for player 3. Then, for player 1 $q$ is optimal. Then for player 2 both $c$ and $d$ can be optimal. For player 1 $e$ is optimal for both $c$ and $d$. Thus we have 2 profiles: $(eq, c, n)$ and $(eq, d, n)$. But for $(eq, c, n)$, it must be $p = 1$ and optimal action for player 2 is $d$, not $c$. For $(eq, d, n)$, it must be $q = 1$ and optimal action for player 3 is $u$, not $n$. Thus, in both profiles there is a player who is not sequentially rational. So no such PBE!

This was exhaustive, thus the set of pure strategy PBE is **empty**.
(b) (6 pts) Find the set of pure strategy Subgame Perfect Nash equilibrium in which player 3 plays $u$. Are they also Perfect Bayesian equilibrium?

**ANSWER:**
There is only one subgame: the game itself. When player 3 plays $u$ we have the following game between 1 and 2:

<table>
<thead>
<tr>
<th></th>
<th>$c$</th>
<th>$d$</th>
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</thead>
<tbody>
<tr>
<td>$eq$</td>
<td>4,3</td>
<td>0,1</td>
</tr>
<tr>
<td>$fq$</td>
<td>2,5</td>
<td>0,1</td>
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<tr>
<td>$zq$</td>
<td>2,2</td>
<td>2,2</td>
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<tr>
<td>$er$</td>
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<td>$fr$</td>
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<tr>
<td>$zr$</td>
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There are 4 Nash equilibrium of this game above where player 3 is fixed to play $u$: $(eq,c),(zq,d),(er,c)$ and $(zr,d)$. Now we have to check whether $u$ is a best response to these strategies of player 1 and 2. In fact, for player 3, both $u$ and $n$ are best responses to all four strategy combinations. Thus, there are four pure strategy SPNE where player 3 plays $u$: $(eq,c,u),(zq,d,u),(er,c,u)$ and $(zr,d,u)$. 
(c) (4 pts) Is there any strategy profile which satisfies the first three requirements of Perfect Bayesian equilibrium (R1,R2 and R3), but not a Perfect Bayesian equilibrium?

**ANSWER:**

Yes. Consider \((e_r, c, u)\) with probabilities \(p = 1\) and \(q = 1\). With \(q = 1\) now it’s optimal for player 3 to play \(u\), and it satisfies R1,R2 and R3. But it is not a PBE. Note that without R4, you can pick any probability at that information set.
4. (24 pts) Based on the evidence in the crime scene, Behzat Ç. and his team start to chase the notorious criminal Ercüment Çözer to arrest as a suspect. However, Ercüment Çözer is either protected by some very powerful bureaucrat or not protected at all, with probabilities $p$ and $1 - p$, respectively. When Behzat's team is about to corner Ercüment Çözer, Ercüment makes a phone call and asks for either a helicopter or a van to come and pick him up. Then, in a few minutes whichever he asked for comes and picks him up. Behzat's team, after observing what picked him up, either keeps chasing him or stops chasing. The payoffs for Behzat are as follows: Behzat gets 0 whenever he stops chasing. He gets -2 whenever he keeps chasing and Ercüment is in a helicopter. He gets 1 if he keeps chasing and Ercüment is in a van and protected. He gets 2 if he keeps chasing and Ercüment is in a van and not protected. The payoffs for ERcüment are as follows: If Ercüment is protected and asks for a helicopter, he gets a payoff of 3 regardless of Behzat's action. If Ercüment is not protected and asks for a helicopter, he gets a payoff of 1 regardless of Behzat's action. If Ercüment is protected and asks for a van, then he gets 3 if Behzat stops chasing, and gets -1 if Behzat keeps chasing. If Ercüment is not protected and asks for a van, then he gets 2 if Behzat stops chasing, and gets -2 if Behzat keeps chasing.

(a) (4 pts) Draw the extensive form game.

**ANSWER:**

![Game Diagram](image-url)
(b) (10 pts) For any given $p$, find the set of pure strategy pooling Perfect Bayesian equilibrium.

**ANSWER:**

First note that regardless of the beliefs, at the left hand info set, $S$ (stop chasing) is optimal, and on the right hand info set $K$ (keep chasing) is optimal.

Fix $(\text{Helicopter, Helicopter})$. Then, $r = p$ and for any $p$, $S$ (stop chasing) is optimal for Behzat on the left hand info set. On the right hand info set, for Behzat, $K$ (keep chasing) strictly dominates $S$ for any $q$. There is no deviation by any of the two types. Thus we have a pooling PBE:

$$((\text{Helicopter, Helicopter}), (\text{S on the left, K on the right}); r = p, \text{ any } q).$$

Fix $(\text{Van, Van})$. Then, $q = p$ and for any $p$, $K$ is optimal for Behzat on the left hand info set. Again, on the right hand info set, for Behzat, $K$ strictly dominates $S$ for any $q$. But then, Protected type deviates to Helicopter to get 3 instead of -1. Note that not protected type also deviates. So, no such PBE.
(c) (10 pts) For any given $p$, find the set of pure strategy separating Perfect Bayesian equilibrium.

**ANSWER:**

Fix $(\text{Helicopter}, \text{Van})$. Then, $r = 1$ and $q = 0$. Behzat picks $S$ on the left, $K$ on the right. But then Not Protected type deviates to Helicopter.

Fix $(\text{Van}, \text{Helicopter})$. Then, $r = 0$ and $q = 1$. Behzat picks $S$ on the left, $K$ on the right. But then Protected type deviates to Helicopter.

Thus, there is no pure strategy separating PBE!
5. (16 pts) Hayalet (H) and Akbaba (A), the two other guys from Behzat’s team, are two roommates sharing an apartment. They play the following game for $T$ periods. Assume that both Hayalet and Akbaba have the same discount factor $\delta \in (0, 1)$.

$$(H)/(A) \quad X \quad Y \quad Z$$

$\begin{array}{ccc}
A & 1,4 & 1,3 & 9,2 \\
B & 0,0 & 5,4 & 0,0 \\
C & 1,1 & 0,0 & 8,8 \\
\end{array}$$

(a) (4 pts) Suppose $T = 2$. How many subgames are there in the repeated game? How many strategies does Akbaba have?

**ANSWER:**

There are 10 subgames and Akbaba has $3^{10}$ strategies.

(b) (12 pts) Suppose now $T = \infty$, that is the game is infinitely repeated. Consider the following grim-trigger strategies:

For Hayalet: At $t = 1$, play C. At any $t > 1$, play C if (C,Z) has been played in all of the previous periods, and play A otherwise.

For Akbaba: At $t = 1$, play Z. At any $t > 1$, play Z if (C,Z) has been played in all of the previous periods, and play X otherwise.

For what values of $\delta$ do these strategies constitute a Subgame Perfect Nash equilibrium?

**ANSWER:**

For those subgames starting with (C,Z), for player 1, the one-shot deviation brings $9 + 1 \cdot \frac{\delta}{1-\delta}$. Sticking to the strategy above brings $8 + \frac{1}{8}$. Thus we need $\delta \geq \frac{1}{8}$ for player 1 not to deviate. For player 2, there is no one shot deviation, no other payoff is larger than 8, which is her payoff at (C,Z).

For those subgames starting with (A,X), there is no possible deviation for any player: For player 1, at (A,X) deviating to B or C brings at most 1, but at (A,X) player 1 gets 1 as well. For player 2, at (A,X) payoff is 4, deviating to Y or Z brings less, either 3 or 2.

Thus, the given strategy profile is a SPNE as long as $\delta \geq 1/8$. 