1. For each game in question 5 in PS6 (Games 1,2,3,4), find the set of perfect Bayesian equilibria in pure strategies only.

**ANSWER:**

**Game 1:** Observe that at the information set of player 2, \(d\) is optimal regardless of the decision node the game reaches. Given \(d\), player 1 picks \(n\) and then \(d\). Thus PBE = \{\((nb, d; p = 1)\)\} where \(p\) is player 2’s belief for the event the game reaches the decision node on the left.

**Game 2:** You can check the Nash equilibria you found. At player 2’s info set, use \(p\) for left hand side decision node, for player 3’s info set use \(q\) for the decision node on the left hand side and \(r\) for the one in the middle. \((a, d, e)\) with beliefs, \(p = 1, q = 1\) is a PBE. Also \((b, d, f)\) with beliefs \(p = 0, q = 0, r = 0\) is another PBE. Finally for \((a, c, f)\), Note that 2’s beliefs must put probability 1 on the node where 1 played a, thus \(p = 1\). 3’s beliefs must have \(r = 1\), but then \(e\) is the optimal action. So this is not a PBE. Thus, PBE = \{\((a, d, e; p = 1, q = 1), (b, d, f; p = 0, q = 0, r = 0)\)\}

**Game 3:** Sequential rationality for 1 at her information set in the left-hand subgame doesn’t pin down his action: depending on her beliefs, she could play either \(g\) or \(h\) here. However, for any beliefs, the only sequentially rational strategy at his information in the subgame on the right is \(i\). So first suppose 1 plays \(g\) in the left-hand subgame and \(i\) in the right. Then for sequential rationality, 2 must play \(c\) in the game on the left and \(e\) in the game on the right. Given this, 1 must play \(b\) at her initial information set. At 1’s information set on the left, assign 1 at the node on the left and 0 at the node on the right in that information set. Note that 1 is sequentially rational playing \(g\). Since 1’s information set on the right is reached, we have to assign 1 to the node on the left there and 0 to the node on the right. With these beliefs, we have sequential rationality for all players. Hence \((bgi, ce; p = 1, q = 1)\) is a perfect Bayesian equilibrium. Next, suppose 1 plays \(h\) in the left-hand subgame (and \(i\), as she must, in the right-hand subgame). For sequential rationality, 2 must play \(d\) and \(e\). Given this, at her initial information set, 1 must play \(b\). Assigning beliefs \(p = 0\) at the unreached information set and \(q = 1\), player 1 is sequentially rational. Hence \((bhi, de; p = 0, q = 1)\) is another perfect Bayesian equilibrium. PBE = \{\((bgi, ce; p = 1, q = 1), (bhi, de; p = 0, q = 1)\)\}.

**Game 4:** The two Nash equilibria are \((b, d, e)\) and \((b, c, f)\). Both are perfect Bayesian equilibria. To see this, note that 2’s beliefs must put probability 1 on the right-hand node in his information set in either equilibrium. For \((b, d, e)\), 3’s beliefs must put probability 1 on the right-hand node in her information set. Given these beliefs, sequential rationality requires 3 to play \(e\). Given this strategy by 3 and 2’s beliefs, sequential rationality requires 2 to play \(d\). Hence this is a perfect Bayesian equilibrium. For \((b, c, f)\), let 3’s beliefs put probability 1 on the left-hand node in his information set (Note that the beliefs cannot be pinned down by the strategies). Given these beliefs, player 3 is
sequentially rational playing $f$. Given this and 2's beliefs, 2 faces a choice between 1 and -1, so player 2 is sequentially rational if she plays $c$. Hence, again, this is a perfect Bayesian equilibrium. Thus \[ \text{PBE} = \{(b, d, e); p = 0, q = 1\}. \]

2. Consider the game below.

(a) Write down the strategy set for each player.

**ANSWER:** \( S_1 = \{LA, LB, MA, MB, RA, RB\} \) and \( S_2 = \{UX, UY, DX, DY\} \)

(b) Find all pure strategy Subgame Perfect Nash equilibria.

**ANSWER:** The only subgame is the game itself. Thus the set of SPNE coincides with the set of Nash equilibria. Transforming the game into the normal form and solving for Nash equilibria we get \( \text{SPNE} = \{(MA, UY), (MA, DY), (RA, UX), (RA, DX), (RB, UX), (RB, DX)\} \)

(c) Find all pure strategy Perfect Bayesian equilibria.

**ANSWER:** Attach \((p, 1 - p)\) to the player 2's information set. And attach \((q, r, 1 - q - r)\) to the player 1's information set.

Fix A for player 1. Then, for \(p = 0\) both X and Y are optimal, and for all \(p > 0\) Y is optimal.

Fix X for player 2, then \(p=0\). Note U is optimal given A. Then given UX and A, player 1 plays R. Thus we have (RA,UX) with \(p = 0\). Note \(r=1\) by Requirement 4. But then player 1, playing A, is not sequentially rational. Now fix Y. Then, any \(p\) can be picked. Given UY and A, player 1 picks M. Thus we have (MA, UY). Thus now \(p\) must be 1 and it must be that \(q = r = 0\). With these beliefs player 1 is sequentially rational playing A. And player 2 is also sequentially rational playing Y. Thus (MA, UY) with \(p = 1\) and \(q = r = 0\) is a PBE.
Now Fix B for player 1. Then Y is optimal only if \( p = 0 \), and X is optimal for any \( p \). Also note that D is optimal.

Fix Y for player 2. Then we have DY and B. Then player 1 picks M. Thus we get (MB,DY). The probabilities must then be \( p = 1 \) and \( q = r = 0 \). But for Y to be optimal \( p \) must be 0. So this is not a PBE. Now fix X for player 2. So we have DX and B. Then player 1 picks R. So we get (RB,DX). The probabilities are \( p = 0 \) and \( q = 1 \). Both players are sequentially rational under these beliefs. Thus this a PBE.

So, \( \text{PBE} = \{((M,A,UY), p = 1, q = r = 0), ((RB,DX), p = 0, r = 1)\} \).

(d) Is there any pure strategy Subgame Perfect Nash equilibrium that is not a Perfect Bayesian equilibrium?

**Answer:** Yes. (MA,UY),(RA,UX),(RA,DX) and (RB,UX) are SPNE but not PBE!