1. Consider the game below.

(a) How many strategies does each player have?

**ANSWER:** Player 1 has 24, player 2 has 24, and player 3 has 6 strategies.

(b) Find the set of pure strategy equilibria using backward induction.

**ANSWER:** Two equilibria: \((zDB, bfRM, Rn)\) and \((zDB, bfRN, Rn)\)

2. There are two plates of ice cream. The small plate contains 2 spoons of ice cream and the big plate contains 6 spoons of ice cream. Two friends, Bonnie and Clyde, have to decide who will get which plate. They both love ice cream and a player’s payoff is the number of spoons he/she eats. They play the following game: First, Bonnie offers Clyde one of the plates, keeping the other plate for herself. If Clyde accepts, each takes his/her agreed plate and eats the ice cream in it, and the game ends. If Clyde rejects, half of the ice cream in each plate suddenly melts (it’s a hot day!). Then, Clyde offers Bonnie one of the plates, keeping the other plate for himself. If Bonnie accepts, each takes his/her agreed plate and eats the ice cream in it, and the game ends. If Bonnie rejects, the rest of the ice cream melts and they are left with no ice cream, and the game ends.
(a) Draw the extensive form game.

\[ \text{ANSWER:} \]

\[ \begin{align*}
\text{B} & \quad \text{C} \\
\text{C} & \quad \text{C} \\
\text{B} & \quad \text{B} \\
\text{B} & \quad \text{B} \quad \text{B} \\
2 & \quad 6 \\
\text{A} & \quad \text{R} \quad \text{R} \\
1 & \quad 3 \quad 1 \quad 3 \\
\text{A} & \quad \text{R} \quad \text{A} \quad \text{R} \quad \text{A} \\
3 & \quad 1 \quad 3 \\
\text{A} & \quad \text{R} \quad \text{A} \quad \text{R} \quad \text{A} \\
0 & \quad 0 \quad 0 \quad 0 \\
\text{B} & \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \\
2 & \quad 6 \\
\end{align*} \]

(b) Write the pure strategy sets for both players.

\[ \text{ANSWER:} \]

\[ S_B = \{x_0x_1x_2x_3x_4 : x_0 \in \{2, 6\}, x_i \in \{A, R\}, i = 1, 4\} = \{2, 6\} \times \{A, R\} \times \{A, R\} \times \{A, R\} \times \{A, R\}. \]

Note that Bonnie has \(2^5 = 32\) strategies.

\[ S_C = \{y_1y_2y_3y_4 : y_1, y_2 \in \{A, R\}, y_3, y_4 \in \{1, 3\}\} = \{A, R\} \times \{A, R\} \times \{1, 3\} \times \{1, 3\}. \]

Note that Clyde has \(2^4 = 16\) strategies.

(c) Find the pure strategy equilibrium by backward induction.

\[ \text{ANSWER:} \]

The equilibrium is \(s^* = ((6\text{AAAA}), (RA11))\). That is, Bonnie offers 6 in the first decision node, and Accepts in all other decision nodes. Clyde Rejects if 2 is offered, and Accepts if 6 is offered, offers 1 in the other two decision nodes. Note that the outcome is "Bonnie offers 6 and Clyde accepts", that is, \((6, A)\).

3. Consider a sequential version of the Cournot quantity setting game. More precisely, assume that firm 1 picks its quantity \(q_1\) first, then firm 2 observes firm 1's quantity decision, and then picks its own quantity \(q_2\). Suppose the market demand is given by \(P = a - Q\), where \(Q = q_1 + q_2\), and \(a\) is a positive number. Each firm has a constant marginal cost, given by \(c\), and has no fixed cost. (This game is called Stackelberg competition in the literature.) Find the equilibrium quantities by backward induction.

\[ \text{ANSWER:} \]

Solving backwards, we first look at firm 2's decision. Firm 2 will maximize

\[ \max_{q_2} [a - (q_1 + q_2) - c]q_2 \]

where the FOC yields the best response

\[ q_2(q_1) = \frac{a - c - q_1}{2} \]

Firm 1 will foresee this and maximize its own profit by taking the best response of firm 2 as given. That is,

\[ \max_{q_1} [a - (q_1 + q_2(q_1)) - c]q_1 = [a - (q_1 + \frac{a - c - q_1}{2}) - c]q_1 = \frac{a - c - q_1}{2}q_1 \]
We obtain \( q_1 = \frac{a-c}{2} \) and plugging this back into \( q_2(q_1) \), we get \( q_2 = \frac{a-c}{4} \).

Note that, the equilibrium is \((q_1^* = \frac{a-c}{2}, q_2^*(q_1) = \frac{a-c-q_1}{2})\), where firm 2 specifies a quantity for each information set (decision nodes which correspond to different levels of \( q_1 \)). The quantities \( q_1 = \frac{a-c}{2} \) and \( q_2 = \frac{a-c}{4} \) constitute the outcome.

4. Suppose there are three firms operating in a market with inverse demand function given by \( P(Q) = 120 - Q \) where \( Q = q_1 + q_2 + q_3 \), and \( q_i \) is the quantity produced by firm \( i \). Each firm has constant marginal cost equal to 12, and no fixed cost. The firms choose their quantities as follows: First, firm 1 chooses its quantity \( q_1 \); then firms 2 and 3 observe \( q_1 \) and then they simultaneously choose their quantities, \( q_2 \) and \( q_3 \). Find the unique pure strategy SPNE.

**ANSWER:** First note that a strategy for firm 1 is \( q_1 \), a non-negative number. A strategy for firm 2 and 3 is a function of \( q_1 \). For every \( q_1 \), there is a subgame, that starts with \( q_1 \), and another subgame which is the game itself. First solve the subgame that starts at \( q_1 \): Given \( q_1 \), the game between 2 and 3 is a Cournot game with \( P(q_2, q_3) = 120 - q_1 - (q_2 + q_3) \). The solution is \( q_2^*(q_1) = q_3^*(q_1) = \frac{108-q_1}{3} \). Now you can use this in the decision of firm 1. Firm 1 maximizes \( \pi_1 = (120 - q_1 - q_2(q_1) - q_3(q_1) - 12)q_1 \) which is \( \pi_1 = (108 - q_1 - 2\frac{108-q_1}{3})q_1 = \frac{108-q_1}{3}q_1 \). The first order condition yields \( q_1^* = 54 \). Thus, the unique SPNE is \((q_1^* = 54, q_2^*(q_1) = q_3^*(q_1) = \frac{108-q_1}{3})\).

5. There is a murder committed in a house and the detective (player 1) suspects the butler (player 2) of the house and engages in evidence collection. However, evidence collection is a random process, and concrete evidence will be available to the detective only with probability 1/2. The butler knows the evidence generating process, but does not know whether the detective received evidence or not. The game proceeds as follows: The detective realizes if he has evidence or not, and then can choose his action, whether to Accuse the butler (A), or Bounce the case (B) and forget it. Once accused, the butler has two options: he can either Confess (C) or Deny (D). Payoffs are realized as follows: If the detective bounces the case then both players get 0 utils. If the detective accuses the butler, and the butler confesses, the detective gains 2 utils and the butler loses 2 utils. If the detective accuses the butler and the butler denies, then payoffs depend on the evidence: If the detective has no evidence then he loses face which is losing 4 utils, while the butler gains glory which gives him 4 utils. If, however, the detective has evidence then he is triumphant and gains 4 utils, while the butler is put in jail and loses 4 utils. Draw the extensive form game, and find the set of SPNE, pure or mixed.

**ANSWER:**

![Extensive Form Game](image)

Nature

\[ \begin{array}{c}
\text{Evidence} \quad 1/2 \\
\text{No evidence} \quad 1/2
\end{array} \]

\[ \begin{array}{c}
\text{Player1} \\
\text{B} \\
\text{A}
\end{array} \]

\[ \begin{array}{c}
\text{Player2} \\
\text{C} \\
\text{D}
\end{array} \]

\[ \begin{array}{c}
\text{Player1} \\
\text{B} \\
\text{A}
\end{array} \]

\[ \begin{array}{c}
\text{Player2} \\
\text{C} \\
\text{D}
\end{array} \]

\[ \begin{array}{c}
\text{Nature} \\
\text{Evidence} \quad 1/2 \\
\text{No evidence} \quad 1/2
\end{array} \]

\[ \begin{array}{c}
\text{Player1} \\
\text{B} \\
\text{A}
\end{array} \]

\[ \begin{array}{c}
\text{Player2} \\
\text{C} \\
\text{D}
\end{array} \]

\[ \begin{array}{c}
\text{Player1} \\
\text{B} \\
\text{A}
\end{array} \]

\[ \begin{array}{c}
\text{Player2} \\
\text{C} \\
\text{D}
\end{array} \]

\[ \begin{array}{c}
\text{Nature} \\
\text{Evidence} \quad 1/2 \\
\text{No evidence} \quad 1/2
\end{array} \]
The strategy spaces are \( S_1 = \{AA, AB, BA, BB\} \) and \( S_2 = \{C, D\} \). The normal form is

\[
\begin{array}{ccc}
1/2 & C & D \\
AA & 2, -2 & 0, 0 \\
AB & 1, -1 & 2, -2 \\
BA & 1, -1 & -2, 2 \\
BB & 0, 0 & 0, 0 \\
\end{array}
\]

There is only one subgame, the game itself. So \( \text{NE} = \text{SPNE} \). There is no pure strategy Nash equilibrium, thus no pure SPNE. Let’s look at mixed SPNE: Since AA strictly dominates BA, and AB strictly dominates BB, we get the reduced game to be:

\[
\begin{array}{ccc}
1/2 & C(q) & D(1 - q) \\
AA(p) & 2, -2 & 0, 0 \\
AB(1 - p) & 1, -1 & 2, -2 \\
\end{array}
\]

The only mixed strategy Nash equilibrium in the reduced game is given by \( p = 1/3 \) and \( q = 2/3 \). Thus \( \text{SPNE} = \{ (AA \text{ with prob } 1/3, AB \text{ with prob } 2/3; C \text{ with prob } 2/3, D \text{ with prob } 1/3) \} \)

6. For each of the game below find the set of SPNE, in pure strategies only.

**Game 1:**

**Answer:** There are two subgames: the game itself and the one that starts with player 1 choosing between b and c. Solving the one that starts with player 1 choosing between b and c, we get (b,d), the only NE of this subgame. Then using the payoffs (3,2) and solving for player 1’s decision between a and n, we see n is optimal. Thus \( \text{SPNE} = \{ nb, d \} \).
Game 2:

**ANSWER:** The only subgame is the game itself. Solving the normal form game

\[
\begin{array}{ccc|ccc}
 & c & d & & c & d \\
 a & 2, 2, 2 & 4, 3, 1 & a & 2, 2, 2 & 0, 0, 0 \\
b & 3, 0, 1 & 3, 1, 0 & b & 0, 0, 0 & 1, 0, 1 \\
 e & & & e & & & f \\
\end{array}
\]

we get three Nash equilibria, \((a, d, e)\), \((b, d, f)\), and \((a, c, f)\), which are all SPNE.

Thus, \(\text{SPNE} = \{(a, d, e), (b, d, f), (a, c, f)\}\).
**Game 3:**

**ANSWER:** There are three subgames: the game itself, the subgame starting after 1’s choice of $a$, and the subgame starting after 1’s choice of $b$. The normal form for the subgame starting after 1 picks $a$ has two pure strategy Nash equilibria: $(g, c)$ and $(h, d)$. The normal form for the subgame starting after 1 picks $b$ has only one Nash equilibrium, $(i, e)$. To continue working backward then, consider 1’s choice at the initial node when $(g, c)$ is the equilibrium that will be played in the subgame if she plays $a$. Clearly, she chooses $b$. Hence one subgame perfect equilibrium is $(bgi, ce)$. The other possibility is that the equilibrium which is played in the subgame following her choice of $a$ is $(h, d)$. In this case, she chooses $b$. Hence the only other pure strategy subgame perfect equilibrium is $(bhi, de)$. Thus, $\text{SPNE} = \{(bgi, ce), (bhi, de)\}$. 
Game 4:

**ANSWER:**

The only subgame is itself. The normal form is

\[
\begin{array}{ccc}
    & c & d \\
\hline
    a & 1, 2, 0 & -1, 1, 0 \\
b & 2, 1, 3 & 0, 3, 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
    & c & d \\
\hline
    a & 1, 2, 0 & 0, 3, 2 \\
b & 2, 1, 3 & 1, -1, 0 \\
\end{array}
\]

Note that 1 has a strictly dominant strategy of b. Hence she must play this in every Nash equilibrium. There are two Nash equilibria in pure strategies: (b, d, e) and (b, c, f). Because there is only one subgame, all are subgame perfect: SPNE= \{(b, d, e),(b, c, f)\}. 