Technological Progress

Some Definitions

A capital saving technological progress (or invention) allows producers to produce the same amount with relatively less capital input.

A labor saving technological progress (or invention) allows producers to produce the same amount with relatively less labor input.

A neutral technological progress allows producers to produce more with same capital labor ratio (do not save relatively more of either input)

i) "Hicks neutral" : Ratio of marginal products remain the same for a given capital labor ratio. Hicks neutrality implies the production function can be written as:
\[ Y = T(t)F(K, L) \]

ii) "Harrod neutral": relative input shares \( K.F_k/LF_L \) remain the same for a given capital output ratio

Harrod neutrality implies the production function can be written as:

\[ Y = F[K, LT(t)] \quad \text{(labor-augmenting form)} \]

where \( T(t) \) is the index of the technology and \( T(t) > 0 \)

labor-augmenting: it raises output in the same way as an increase in the stock of labor.

iii) "Solow neutral": relative input shares \( LF_L/K.F_k \) remain the same for a given labor output ratio
Solow neutrality implies the production function can be written as:

\[ Y = F[KT(t), L] \] (capital-augmenting form)
Solow Model with labor augmenting technological progress

Suppose the technology $T(t)$ grows at rate $x$

$$\dot{K}(t) = I(t) - \delta K(t) = sF(K(t), T(t)L(t)) - \delta K(t)$$  \hfill (29)

Dividing by $L(t)$

$$\dot{k} = sF(k, T(t)) - (n + \delta)k \quad \text{and} \quad \frac{\dot{k}}{k} = s\frac{F(k, T(t))}{k} - (n + \delta)$$

The average product of per capita capital $\frac{F(k, T(t))}{k}$ now increases over time because the $T(t)$ grows at a rate $x$.

Steady state growth rate: By definition the steady state growth rate $\left(\frac{\dot{k}}{k}\right)^*$ is constant

$$s\frac{F(k^*, T(t))}{k^*} - (n + \delta) = \text{constant}$$
Since $F$ is CRS, $\frac{F(k, T(t))}{k} = F(1, \frac{T(t)}{k})$. This implies that $T(t)$ and $k$ grow at the same rate $x$, because $s, n$ and $\delta$ are constants

\[
\left( \frac{k}{\bar{k}} \right)^* = x
\]

Moreover since $y = F(k, T(t)) = kF(1, \frac{T(t)}{k})$

\[
\left( \frac{y}{\bar{y}} \right)^* = x
\] (30)

and $c = (1 - s)y$, $\left( \frac{c}{\bar{c}} \right) = \left( \frac{(1-s)y}{(1-s)y} \right) = x$

**Transitional Dynamics**

Define: effective amount of labor$=\text{physical quantity of labor} \times \text{efficiency}$

of labor $= L \times T(t) \equiv \^L$
\[ \hat{k} = \frac{K}{LT(t)} = \frac{k}{T(t)} = \text{capital per unit of effective labor.} \]

\[ \hat{y} = \frac{Y}{LT(t)} = F(\hat{k}, 1) = f(\hat{k}) = \text{output per unit effective labor} \]

We can rewrite

\[ \dot{K} = sF(K, T(t)L) - \delta K \quad \text{divide both sides by } T(t)L \]

\[ \frac{\dot{K}}{T(t)L} = sf(\hat{k}) - \delta \hat{k} \quad \text{(31)} \]

\[ \hat{k} = \left( \frac{K}{T(t)L} \right) = \frac{LT(t)\dot{K} - K(\dot{LT(t)} + LT(\dot{t}))}{T(t)^2L^2} = \frac{\dot{K}}{T(t)L} - \frac{\dot{kn}}{T(t)L} - \frac{\dot{kx}}{T(t)L} \]

Therefore

\[ \frac{\dot{K}}{T(t)L} = \hat{k} + \frac{\dot{kn}}{T(t)L} + \frac{\dot{kx}}{T(t)L} \]

substituting in (31)
\[
\hat{k} = s f(k) - (x + n + \delta) \hat{k}
\]  \quad (32)

and

\[
\frac{\hat{k}}{k} = s \frac{f(k)}{k} - (x + n + \delta)
\]  \quad (32)

where \(x + n + \delta\) is the effective depreciation rate

The effective per capita capital depreciates at the rate \(x + n + \delta\)
Behavior of the growth rate in the Solow Model with labor augmenting technological progress (1)
Speed of Convergence:

The speed of convergence is given by
For the CD production function

\[ \frac{\dot{k}}{k} = sA\lambda\kappa - (1 - \alpha) - (x + n + \delta) \]

or

\[ \frac{\dot{k}}{k} = sAe^{-\lambda\log(k)} - (x + n + \delta) \]

\[ \beta = (1 - \alpha)sA\lambda\kappa - (1 - \alpha) \quad \text{(declines monotonically)} \]

Near the steady state

\[ sA\lambda\kappa - (1 - \alpha) = (x + n + \delta) \]
\[ \beta^* = (1 - \alpha)(x + n + \delta) \]  \hspace{1cm} (34)

**What does the data say about convergence?**

Consider the benchmark case with
\[ x = 0.02, \; n = 0.01 \; \text{and} \; \delta = 0.05 \; \text{(for US)} \]

where \( x \) is the long term growth rate of GDP per capita

\[ \beta^* = (1 - \alpha)(x + n + \delta) = (1 - \alpha)(0.08) \]

which depends on \( \alpha \)

Suppose \( \alpha = 1/3 \; (\text{based on data}) \) then \( \beta^* = 5.6\% \) (half life of 12.5 years)

But the data says that \( \beta^* \simeq 2 - 3\% \) which implies \( \alpha = 3/4 \) (too high for physical capital)

-A broader definition of capital is needed to reconcile theory with the
facts
Extended Solow Model with human capital

\[ Y = AK^\alpha H^\eta [T(t)L]^{1-\alpha-\eta} \text{ and } \dot{y} = \dot{A}k \dot{h} \]  
\[ \dot{k} + \dot{h} = sA^\alpha \dot{h} - (x + n + \delta) \left( \dot{k} + \dot{h} \right) \]  

It must be the case that returns to each type of capital are equal.

\[ \frac{\dot{y}}{k} = \frac{\dot{y}}{h} - \delta \text{ and } \dot{h} = \eta \dot{k} \]

Using in (36)

\[ \dot{k} = sA^\alpha - (\delta + n + x) \text{ where } \tilde{A} = \text{constant} \]

\[ \beta^* = (1 - \alpha - \eta)(x + n + \delta) \]
Now with $\alpha = 1/3$ (based on data) then $\beta^* = 2.1\%$ (a better match)
What’s Wrong with Neoclassical Theory??
- Does not explain long-term consistent per capita growth rates.
- Can not maintain perfect competition assumption when technological progress is not exogenous.

The AK Model

\[ Y = AK \]
\[ \frac{k}{k} = sA - (n + \delta) > 0 \text{ for all } k \text{ if } sA > (n + \delta) \]

does not exhibit conditional convergence. How?

How about

\[ Y = AK + BK^\alpha L^{1-\alpha} \text{ where } A > 0, B > 0 \text{ and } 0 < \alpha < 1 \]
Constant Elasticity of Substitution (CES) Production Functions

\[ y = F(K, L) = A \left\{ a(bK)\psi + (1 - a) [(1 - b)L]^{\psi} \right\}^{\frac{1}{\psi}} \]

0 < \( a \) < 1

0 < \( b \) < 1

and

\[ \psi < 1 \]

The elasticity of substitution is a measure of the curvature of the isoquants where the slope of an isoquant is given by

\[ \frac{dL}{dK} = -\frac{\partial F/\partial K}{\partial F/\partial L} \]

\[ \left[ \frac{\partial (slope) L/K}{\partial (L/K) Slope} \right]^{-1} = \frac{1}{1-\psi} \]

Properties of CES production function
1) The elasticity of substitution between capital and labor, $\frac{1}{1-\psi}$, is constant

2) CRS for all values of $\psi$

3) As $\psi \to -\infty$, the production function approaches $Y = \min \{bK, (1 - b)L\}$.

As $\psi \to 0$, $Y = (\text{constant})K^aL^{1-a}$ (CD)

For $\psi = 1$, $Y = abK + (1 - a)(1 - b)L$ (linear) so that K and L are perfect substitutes (infinite elasticity of substitution)

Proof: Take log of Y and apply L’Hospital’s rule. *i.e. find* $\lim_{\psi \to 0} [\log Y]$ 

Transitional Dynamics with CES production function

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (n + \delta)$$

Note that for CES
\[ f(t(k)) = Aab^\psi \left[ ab^\psi + (1 - a)(1 - b)^\psi k^{-\psi} \right]^{\frac{1-\psi}{\psi}} \]

and

\[ \frac{f(k)}{k} = A \left[ ab^\psi + (1 - a)(1 - b)^\psi k^{-\psi} \right]^{\frac{1}{\psi}} \]

i) \( 0 < \psi < 1 \)

\[ \lim_{k \to \infty} \left[ f'(k) \right] = \lim_{k \to \infty} \left[ \frac{f(k)}{k} \right] = A ab^{\frac{1}{\psi}} > 0 \]

\[ \lim_{k \to 0} \left[ f'(k) \right] = \lim_{k \to 0} \left[ \frac{f(k)}{k} \right] = \infty \]

Therefore, CES can exhibit can generate endogenous growth for \( 0 < \psi < 1 \), if savings rates are high enough.
CES Model with \( 0 < \psi < 1 \) and \( sAba^\psi > \delta + n \)

ii) \( \psi < 0 \)

\[
\lim_{k \to \infty} \left[ f'(k) \right] = \lim_{k \to \infty} \left[ \frac{f(k)}{k} \right] = 0
\]

\[
\lim_{k \to 0} \left[ f'(k) \right] = \lim_{k \to 0} \left[ \frac{f(k)}{k} \right] = Aba^\psi < \infty
\]
No endogenous growth, negative growth rates are possible for low values of $s$
Poverty Traps

for some range of k, the average product of capital is increasing in k
- non-constant savings rates
- increasing returns with learning by doing and spillovers
Asimple model

Suppose the country has access to two technologies

\[ y_A = Ak^\alpha \]
\[ y_B = Bk^\alpha - b \]

where \( B > A \). To employ \( y_B \) gov’t has to incur a setup cost of \( b \) per worker.

Under what conditions will the government incur \( b \)?

\[ y_A \leq y_B \]

or \( Ak^\alpha \leq Bk^\alpha - b \)

and \( k \geq \tilde{k} \) where \( \tilde{k} = \left[ \frac{b}{B - A} \right]^\frac{1}{\alpha} \)

\[ \frac{k}{\tilde{k}} = s\frac{Ak^\alpha}{k} - (n + \delta) \]
\[ \frac{k}{\tilde{k}} = s\frac{Bk^\alpha - b}{k} - (n + \delta) \]