1. \( U(x_1, x_2) = x_1^{1/3} x_2^{2/3} \) \( M = 16 \) \( p = (2, 4) \)

(a) \( \max \ x_1^{1/3} x_2^{2/3} \)

s.t. \( 2x_1 + 4x_2 \leq 36 \)
\( x_1, x_2 \geq 0. \)

(b) \( L(x_1, x_2, \lambda) = x_1^{1/3} x_2^{2/3} + \lambda (36 - 2x_1 - 4x_2) \)

\[ \frac{dL}{dx_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/3} - 2\lambda = 0 \quad (1) \]

\[ \frac{dL}{dx_2} = \frac{2}{3} x_1^{1/3} x_2^{-1/3} - 4\lambda = 0 \quad (2) \]

\[ \frac{dL}{d\lambda} = 2x_1 + 4x_2 - 36 = 0 \quad (3) \]

(c) From (1) and (2), we obtain \( x_1 = x_2. \) Plug in (3),

\[ 6x_1 = 36 \Rightarrow \frac{x_1}{6} = \frac{x_2}{6} \]

(2) \( U(x_1, x_2) = x_1^{3/2} x_2 \) \( M = 100 \) \( p = (3, 4) \)

(a) \( \max \ x_1^{3/2} x_2 \)

s.t. \( 3x_1 + 4x_2 \leq 100 \)
\( x_1, x_2 \geq 0. \)

(b) \( L(x_1, x_2, \lambda) = x_1^{3/2} x_2 + \lambda (100 - 3x_1 - 4x_2) \)

\[ \frac{dL}{dx_1} = \frac{3}{2} x_1^{1/2} x_2 + 3\lambda = 0 \]
\[ \frac{dL}{dx_2} = x_1^{3/2} - 4 \lambda = 0 \]
\[ \frac{dL}{d\lambda} = 100 - 3x_1 - 4x_2 = 0. \]

\[ x_1^{3/2} = 4\lambda \quad \text{and} \quad x_1^{1/2}x_2 = 2\lambda \quad \Rightarrow \quad x_1^{3/2} = 2x_1^{1/2}x_2 \]
\[ (x_1 = 2x_2) \quad \Rightarrow \]

plug into budget constraints:

\[ 100 - 3(2x_2) - 4x_2 = 0 \]

\[ x_2^* = 10 \quad x_4^* = 20 \]

(3) For a given utility level \( u = 20 \), consider \( x_2 + 2x_4 = 20 \) and \( x_1 + 2x_2 = 20 \). Solving for \( x_2 \), we obtain \( x_2 = 20 - 2x_4 \) and \( x_2 = 10 - \frac{x_1}{2} \).

We plot these two lines in the picture below.

The indifference curve is the northeast boundary of these two lines.

In particular, for bundle \((x_1, x_2) = (1, 18)\), located at point A in the figure, the consumer's utility is

\[ \min \{18 + 1, 1 + 18\} = \min \{20, 37\} = 20. \]

Similarly, bundle B in the other extreme of the figure, i.e., \((x_1, x_2) = (16, 2)\) yields a utility level of

\[ \min \{2 + 2, 16 + 2 \cdot 2\} = 20. \]
Note that bundles in the southeast boundary, such as $C = (1, 9.5)$, only provide a utility of $\min \{9.5 + 2.1, 1 + 2.9 \} = 11.5 < 20$, so the southwest boundary of the two lines cannot be the indifference curve of $u = 20$.

\[\frac{p_1}{p_2} \leq 2\]

Similarly, if the budget line is flatter than $1/2$ in absolute value, e.g. $-1/4$, $x_2$ will equal $0$. 

\[\frac{p_1}{p_2} > 2\]
If the optimum is unique, it must occur where at the line $x_2 - 2x_4 = x_1 - 2x_2$. Since line $y_2 = 2x_4$ crosses $x_1 - 2x_2$ at the 45° line, the interior optimum occurs at $x_1 = x_2$, so that $x_1/x_2 = 1$.

The allocation $\mathbf{X}$ is feasible if

$$x_1^1 + x_1^2 + x_1^3 \leq e_1^1 + e_1^2 + e_1^3$$
$$x_2^1 + x_2^2 + x_2^3 \leq e_2^1 + e_2^2 + e_2^3$$
$$x_3^1 + x_3^2 + x_3^3 \leq e_3^1 + e_3^2 + e_3^3.$$ 

a) Not feasible
b) Feasible allocation ($y$)
   - $y^1 = (3, 0, 0)$
   - $y^2 = (0, 0, 0)$
   - $y^3 = (0, 0, 3)$

c) A non-feasible allocation ($z$)
   - $z^1 = (4, 0, 0)$
   - $z^2 = (1, 1, 1)$
   - $z^3 = (3, 3, 3)$.

d) The only Pareto efficient allocation in this economy is the one given in part b. This allocation is in the core. So, there is no P.E allocation which is not in the core.

e) Allocation $y$ provided in part b.
a) \( x^1 = (0, 4) \), \( x^2 = (5, 0) \), \( x^3 = (0, 0) \) is not in the core, since agent 3 can block it using his own endowment.

\[ u^3(2, 1) > u^3(0, 0) \]

b) \( x^1 = (2, 1) \), \( x^2 = (2, 1) \), \( x^3 = (1, 2) \) is not in the core, since agents 1 and 2 can block it. To see this, take allocation where \( y^1 = (0, 3) \), \( y^2 = (2, 0) \), \( y^3 = (2, 1) \)

\[ y^1 > x^1 \]
\[ y^2 > x^2 \]
\[ y^1 + y^2 > e^1 + e^2 \]

(Continued on next page)

6) All utility functions are of Cobb-Douglas form, so we know that

\[ x^1_i = \frac{1}{3} \frac{3p_1}{P_1} \]
\[ x^2_i = \frac{2}{3} \frac{3p_2}{P_2} \]
\[ x^3_i = \frac{1}{2} \frac{2p_1 + p_2}{P_1} \]

using feasibility constraint for good 1

\[ x^1_i + x^2_i + x^3_i \leq e^1_i + e^2_i + e^3_i \]

(This will hold with equality)

\[ \frac{1}{3} \frac{3p_1}{P_1} + \frac{2}{3} \frac{3p_2}{P_2} + \frac{1}{2} \frac{2p_1 + p_2}{P_1} = 5 \]

\[ 1 + \frac{2p_2}{P_1} + 1 + \frac{1}{2} \frac{p_2}{P_1} = 5 \]

\[ \frac{5p_2}{P_1} = 3 \]

\[ \frac{p_2}{P_1} = 6/5 \] or \[ \frac{p_1}{P_2} = \frac{5}{6} \]

Let \( p = (5, 6) \) \( \Rightarrow \) \( x^*_i = (x^1_i, x^2_i, x^3_i) = (4, 1, 2/5, 8/5) \).
Similarly:

\[ x_1^* = \frac{2}{3} \frac{2p_1}{P_2}, \quad x_2^* = \frac{1}{3} \frac{3p_2}{P_2}, \quad x_3^* = \frac{1}{2} \frac{2p_1 + p_2}{P_2} \]

\[ x_2^* = (\frac{10}{6}, 1, \frac{8}{6}) \]

Walrasian eq: \( (p^*, x^*) = (\frac{5}{6}, x_1^*, x_2^*) \)

4) Suppose not. Then we have an allocation where both agents are given positive amounts of both goods and this is P.E. Call this \( x, x^1 = (a, b), x^2 = (c, d) \) and \( a, b, c, d > 0 \).

\[ a + c < e_1^1 + e_1^2 \]
\[ b + d < e_2^1 + e_2^2 \]

\[ \exists \varepsilon > 0 \text{ such that } \varepsilon < a, \varepsilon < c, \varepsilon < b, \varepsilon < d \]

Now take allocation \( y \) where

\[ y^1 = (a - \varepsilon, b + \varepsilon) \]
\[ y^2 = (c + \varepsilon, d - \varepsilon) \]

\[ u^1(y^1) - u^1(x^1) = 2\varepsilon > 0 \Rightarrow y^1 \succ x^1 \]
\[ u^2(y^2) - u^2(x^2) = 3\varepsilon > 0 \Rightarrow y^2 \succ x^2. \]

Hence, \( x \) cannot be Pareto efficient... Contradiction!

8) a) Since there are only 2 agents, the only blocking conditions are the agents themselves. But \( u^1(x^1) > u^2(e^1) \) and \( u^2(x^2) > u^2(e^2) \)

Therefore, no blocking will occur. \( x \) is in the core.
(b) $X$ can be blocked by a coalition of $(1,2,3)$ to see this take allocation $y$ where $y^1 = (6,3)$, $y^2 = (6,6)$, $y^3 = (6,8)$, $y^4 = (2,8)$, $y^1 \not> x^1$, $y^2 \not< x^2$, $y^3 \not< x^3$, $y^1 + y^2 + y^3 \leq e^1 + e^2 + e^3$.

(c) $X$ is not in the Core, since agent 3 can block it using her endowment $u^3(e^3) > u^3(x^3)$.

(9) $X$ is the Walrasian eq. allocation for this economy. The slope of the line which connects e ox gives the relative prices that sustain the Walrasian equilibrium.

(10) Suppose all three consumers envy someone else. Without loss of generality, suppose $a$ envies $b$. If $b$ envies $a$ then exchanging $a$ and $b$'s commodity bundle would make for a Pareto improvement, contradiction.

So $b$ must envy $c$. If $c$ envies $b$ again we contract Pareto optimality of $x$. So $c$ must envy $a$. But now we have a three-way cycle and we can create a Pareto improvement by giving $x_b$ to consumer $a$, $x_c$ to consumer $b$ and $x_a$ to consumer $c$. This proves that there must be at least one who doesn't envy anyone else.
(a) It is not complete. Recall that completeness requires for every pair $x$ and $y$, either $x \succ y$ or $y \succ x$ (or both). To see why this property does not hold, consider two bundles $x, y \in \mathbb{R}^2$ with bundle $x$ containing more units of good 1 than bundle $y$ but fewer units of good 2, i.e., $x_1 > y_1$ for good 1 but $x_2 < y_2$ for good 2. Then we have neither that $x \succ y$ (since for that we would need $x_1 > y_1$ and $x_2 > y_2$) nor $y \succ x$ (i.e., for what we would need $y_1 > x_1$ and $y_2 > x_2$).

(b) It is transitive. Recall that transitivity requires that, for any three bundles $x, y$ and $z$, if $x \succ y$ and $y \succ z$ then $x \succ z$. Now $x \succ y$ and $y \succ z$ means that $x_1 > y_1$ and $y_2 > z_2$.

In vector notation, this means that bundle $x$ is weakly larger than $y$ in every component. Similarly, bundle $y$ is weakly larger than bundle $z$ in every component. Hence $x \succ y$ and $y \succ z \Rightarrow x_1 > y_1$ and $y_2 > z_2$ for every good $e$.

By transitivity of the "greater or equal than" symbol $\succ$, we thus have $x_1 \geq y_1$ for all goods $e$, and so $x \succ z$ (i.e., bundle $x$ is weakly larger than bundle $z$ in every component.) That is $x \succ z$ holds, as required.

(c) It is not strictly monotonic:

Strict monotonicity implies that if $x \succ y$ and $x_j > y_j$ for some $j$, implies $x \succ y$. However, in the question, definition says even $y_j > y_j$ for all $j$, $x \succ y \Rightarrow$ Thus, not strictly monotonic.
Namely, let \( x = (4, 3) \) and \( y = (3, 2) \).

It satisfies this preference relation; however, strict monotonicity implies \( x > y \) for any.

\( \Box \) It is strictly convex. Recall that strict convexity requires if \( x > z \) and \( y > z \), then the linear combination of bundles \( x \) and \( y \),

\[ \alpha x + (1-\alpha) y, \] \( \) is strictly preferred to \( z \), \( \alpha x + (1-\alpha) y > z \) for all \( \alpha \in (0,1) \).

Now if \( x > z \) and \( y > z \), then in the preference relation we are analyzing, it means that bundle \( x \) is weakly larger than bundle \( z \) in every component, and similarly for bundle \( y \), i.e., \( x_k > z_k \) and \( y_k > z_k \) for all goods \( k \). And if bundles \( x \) and \( z \) are different, \( x \neq z \), then for some good \( k \in \{1, \ldots, 2, \ldots, 5 \} \) we must have \( x_k > z_k \). Thus, for any \( \alpha \in (0,1) \), the linear combination of \( x \) and \( y \) lies above bundle \( z \), i.e.,

\[ \alpha x_k + (1-\alpha) y_k > z \] \( \) for all good \( k \), and

\[ \alpha x_k + (1-\alpha) y_k > z_k \] \( \) for some good \( k \).

Hence, we have that \( \alpha x + (1-\alpha) y > z \) and \( \alpha x + (1-\alpha) y \neq z \), and so,

\( \alpha x + (1-\alpha) y \geq z \) and not \( z \geq \alpha x + (1-\alpha) y \). Therefore \( \alpha x + (1-\alpha) y > z \) as required for strict convexity.
\[ \sum_{k=1}^{n} \sum_{i \in I} P_k \left( x_k^i (p, pe^i) - e_k^i \right) = 0. \]

Sum over all individuals.

\[ \sum_{i \in I} \sum_{k=1}^{n} P_k \left( x_k^i (p, pe^i) - e_k^i \right) = 0. \]

Order of summation is immaterial, so we can reverse it and write:

\[ \sum_{k=1}^{n} P_k \left( \sum_{i \in I} x_k^i (p, pe^i) - \sum_{i \in I} e_k^i \right) = 0. \]

This gives:

\[ \sum_{k=1}^{n} P_k \left( \sum_{i \in I} x_k^i (p, pe^i) - \sum_{i \in I} e_k^i \right) = z_k(p) \]

\[ \sum_{k=1}^{n} P_k z_k(p) = 0. \]

Thus, Walras' Law: If aggregate excess demand is 0 in n-1 markets, then it should be also 0 in the n\textsuperscript{th} market.