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Pareto-efficient Rules with Approval Ballotting

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Abstract We consider a situation where a committee has to be chosen from a finite set of candidates. This committee can be either of variable or of fixed size. Voters cast approval ballots, where they approve as many candidates as they want. Approval ballots are used to define voters' preferences over committees, by means of a preference extension rule. For the variable size case, we characterize the largest rich domain of separable and top-consistent preference extension rules for which issue-wise majority voting is Pareto-efficient, i.e., always yields out a Pareto-optimal committee. Furthermore, in the election of a fixed size committee of size k, we characterize the largest rich domain of such preference extension rules for which sequential approval voting (which selects the k first most approved candidates) is Pareto-efficient.

Key words Approval Ballotting - Pareto efficiency - Committees - Majority Rule

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1 Introduction

Given a set of voters and a finite set of candidates, a committee election problem is a problem of electing a subset of the given candidate set. Among a variety of models designed, we study committee elections with approval balloting. The given set of voters cast approval ballots that show which candidates they approve and which they do not. A predetermined voting rule is applied over these approval ballots to determine the election outcome. We distinguish two cases: Electing a committee without any size restriction or electing a fixed-size committee. In the case of committee election without a size restriction, we consider issue-wise majority rule as the voting rule. Hence, the winning committee would be the one that reflects majority will over each candidate. In other words, the committee that consists of the members collecting more approvals than disapprovals would be elected. In the case of a fixed-size committee election, sequential approval

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voting is applied. That is to say, if the winning committee is restricted to be of size k, sequential approval voting elects the k candidates collecting the highest number of approvals.

A natural question to be asked in committee election problems is related to the representativeness of election outcomes. Depending on some hidden preferences, voters cast their approval ballots and the voting rule yields the election outcome. But to comment on how successfull that voting rule is to reflect those hidden preferences requires some inquiry. For instance, in the existence of a pair-wise majority winner, namely, the Condorcet winner, electing it will be consistent with the underlying preferences. Kadane (1972) shows that with multiple binary issues -which is the case studied in this paper- under separable preferences the issue-wise majority rule selects the Condorcet winner, whenever it exists. The essential ingredient in this result is the separability axiom, which refers to the independence of the decision over an issue from the decisions over other issues. This result is later extended by Schwartz (1977) to the context of vote trading and sophisticated voting. He shows that Kadane's result is valid under either sophisticated or sincere or simultaneous or sequential voting. Therefore, Kadane's and Schwartz's results promote the use of issue-wise majority rule as a decisive tool in the settings that separation of issues is reasonably applicable such as committee elections or referendum voting.

In the absence of a Condorcet winner, Condorcet consistent choice concepts such as the Uncovered Set and Top Cycle are referred as representativeness criteria. A recent study shows that even under a very restricted assumption over preferences over outcomes, which is Hamming rule, the issue-wise majority winner can be covered in the majority tournament among outcomes (Laffond and Lainé (2008)). A second result in the same work ensures the inclusion of the issue-wise majority winner to the Top-Cycle. But since McKelvey (1976), Top-Cycle is not considered to be a "reliable" stability concept as it can consist of all of the alternative set. In addition to poor-selectivity, the Top-Cycle may select Pareto-dominated outcomes. This suggests to investigate the Pareto characteristics of voting rules.

As a reasonable minimum representativeness criterion Pareto-optimality prohibits the election of an alternative while another alternative is preferred to it by every individual in the society. We call a voting rule Pareto-efficient if it always yields Pareto-optimal outcomes. Obviously Pareto-efficiency of a voting rule does not ensure full representativeness but the violation of Pareto-efficiency gives a clue about how poorly representative the voting rule can be. Özkal-Sanver and Sanver (2006) shows that, under the assumption of separable preferences, it is impossible to guarantee Pareto optimal outcomes through any kind of anonymous referendum voting, in particular issue-wise majority rule. This result triggers a natural question for our search: Under which conditions the issue-wise majority rule and the sequential approval voting will yield Pareto-optimal outcomes?

Obviously, in addition to separability, ensuring Pareto-optimal outcomes requires more restricted preference domains. Benoit and Kornhauser (1994) study numbered post elections, where each issue refers to a specific position in an assembly or a specific post in a legislation, and assume that the order of the importance of the issues is the same for all voters; for instance, the most important issue to be voted is the presidency position and the second important is the asistant president position, etc. They show that in a numbered post election with a common order of importance of the issues, if the preferences are separable and top-lexicographic then a Pareto-optimal assembly is always selected by issue-wise majority rule. This top-lexicographicity property can be roughly defined as such: Consider two assemblies with the most important position being the first issue for the voters and the second most important one being the second and so on. Preferences would be top-lexicographic if any voter prefers the first assembly to the second where the first issue that these two committees differ is occupied by the voter's best preference for this position in the first assembly. Notice that this top-lexicographicity property is designed for a very special type of voting procedure, which is the numbered post election with a common order of importance of the issues, and does not guarantee Pareto-optimality in more general settings.

In this paper we try to find under which conditions, these specific voting rules that depend on approval balloting are Pareto-efficient. To define voters' preferences over committees, by means of a preference extension rule we extend approval ballots to complete preference orders over committees. For the variable size case, we characterize the largest rich domain of separable and top-consistent preference extension rules for which issue-wise majority voting is Pareto-efficient. Similarly, in the election of a fixed size committee, we characterize the largest rich domain of such preference extension rules for which sequential approval voting is Pareto-efficient.

The paper is organized as follows: The model of committee choice is described in Part 2. Results, for both cases of variable and fixed size, are stated and proven in Part 3. Additional comments are provided in Part 4, together with several suggestions for further research.

2 The Model

A finite set of voters $\mathcal{N} = \{1, ..., n, ...N\}$ has to choose members of a committee from a finite set of candidates $\mathcal{C} = \{1, ..., c, ..., C\}$. This committee can either be of a given fixed size k, or be of any possible size. Each voter n casts an approval ballot, defined as a vector $x_n = (x_n^c)_{c=1,...,C} \in \{0,1\}^C$, where $x_n^c = 1$ (resp. $x_n^c = 0$) means that n approves (resp. disapproves) candidate c.

A *committee* is a subset of $\mathcal{C}$. It should be clear that any committee involving k members can be described as an element $x \in \{0,1\}^C$, where $k =\mid 1(x) = \{c \in \mathcal{C} : x^c = 1\} \mid$. Let $\Omega = \cup_{C \geq 1}\{0,1\}^C$ be the set of all possible committees that can be chosen from any finite set of candidates (including the no-member and the all-member committees). Given an integer k, the subset Ω_{Ck} contains all k-sized committees (i.e. all committees x such that $k =\mid 1(x) \mid$) from a set of Q candidates, and $\Omega_k == \cup_{C \geq 1}\Omega_{Ck}$.

A *ballot set* is a matrix $X^{NC} = [x_n^c]_{n=1,...,N}^{c=1,...,C}$, where row n corresponds to voter n's approval ballot $x_n \in \{0,1\}^C$. Let $\mathcal{X} = \cup_{N,C \geq 1} X^{NC}$.

2.1 Voting rules

A *voting rule* is a correspondence V from $\mathcal{X}$ to Ω, such that, for any $(n.C)$-ballot X^{NC}, $V(X^{NC}) \subseteq \{0,1\}^C$. We focus below on two specific voting rules.

The *issue-wise majority rule* selects from a ballot set X^{NC} a committee whose members are more often approved than disapproved.

Definition 1 *The issue-wise majority rule is the voting rule M defined by: $\forall X^{NC} \in \mathcal{X}$, $M(X^{NC}) = (m^1, ..., m^C) \in \{0,1\}^C$ where $\forall c = 1, ..., C$, $\mid \{n = 1, ..., N : x_n^c = m^c\} \mid \geq \mid \{n = 1, ..., N : x_n^c \neq m^c\} \mid$.*

Note that the outcome of M is unique whenever M is odd. Moreover, M is a specific element of the more general class of *issue-wise quota rules*:

Definition 2 *Let $\alpha \in [\frac{1}{2}, 1]$. The issue-wise α-rule is the voting rule V_α defined by: $\forall X^{NC} \in \mathcal{X}$, $V_\alpha(X^{NC}) = (v_\alpha^1, ..., v_\alpha^C) \in \{0,1\}^C$ where $\forall c = 1, ..., C$, $\mid \{n = 1, ..., N : x_n^c = v_\alpha^c\} \mid \geq \alpha.N$.*

Furthermore, it should be obvious that, when the committee size is restricted, an issue-wise quota rule may fail to select an outcome having the relevant size k (even if all ballots belong to Ω_k). In this case of a fixed sized committee, we focus on the following sequential approval voting rule:

Definition 3 *Let C and k be any two integers such that $k \leq C$. The k-sequential approval voting rule is the voting rule S_k defined by: $\forall X^{NC} \in \mathcal{X}$, $S_k(X^{NC}) = \{s = (s_k^1, ..., s_k^C) \in \Omega_k$ such that $\forall c \in 1(S_k(X^{NC}))$, there is no $c' \notin 1(S_k(X^{NC}))$ such that $\Sigma_{n \in \mathcal{N}}\ (x_n^{c'} - x_n^c) > 0\}$.*

Allowing for any possible fixed committee size k, we call *sequential approval rule* the rule S defined on $\mathcal{X}$ by: $\forall k$, $\forall C > k$, $\forall X^{NC} \in \mathcal{X}$, $S(X^{NC}) = S_k(X^{NC})$.

According to S, voters may approve as many candidates as they like, and any elected committee contains exactly k candidates among those who are the k best ones according to their number of approvals.

2.2 Pareto efficient voting: unrestricted size

Studying how representative are the outcomes of alternative voting rules, and in particular their Pareto efficiency, requires to define the voters' preferences over committees. Since approval ballots are the only observed data, the preferences over committees have to be built from the ballots.

Let us address first the case of unrestricted size. We assume such preferences are complete preorders over $\{0,1\}^C$, and are obtained from ballots by means of a preference extension rule. Formally, let Γ^C the set of all complete preorders on $\{0,1\}^C$. Let $R = \cup_{C \geq 1} \Gamma^C$.

A *preference extension rule* is a function R from Ω to R which associates with each ballot $x \in \{0,1\}^C$ an element $R(x)$ of Γ^C. The asymmetric counterpart of R is denoted by P and I stands for the indifference part. A preference extension rule R associates with each possible ballot x over any number C of candidates, a complete preorder $R(x)$ over the committees built from C candidates. Let $\mathcal{F}$ stand for the set of all possible extension rules. Given any ballot set X^{NC}, one get a preference profile over committees by assigning each of the N voters an extension rule in $\mathcal{F}$. We denote by R_n the extension rule used by voter n, and by $\rho = (R_1, ..., R_N)$ the vector of extension rules prevailing for all voters $n = 1, ..., N$. Any vector ρ maps a ballot set X^{NC} to a preference profile $\rho(X^{NC}) = (R_1(x_1), ..., R_N(x_N)) \in \Gamma^{CN}$. To simplify notations, we will denote the extended preorder from voter n's approval ballot as R_n instead of $R_n(x_n)$.

We impose two properties for a preference extension rule to be admissible, respectively called *top-consistency* and *separability*.

An extension rule R is said to be *top-consistent* if $\forall x \in \Omega$, $x\ P(x)\ y$ for all $y \in \Omega - \{x\}$, that is if each voter's ballot describes her most preferred committee (in other words, each vote is sincere).

In order to define separability, we need first to introduce sub-committees. Let $x \in \{0,1\}^C$ and let $B \subseteq \{1, ..., C\}$ be a subset of C' candidates. The *sub-committee* x/B is the element of $\{0,1\}^{C'}$ defined by:$\forall c \in B$, $(x/B)^c = x^c$. Then the preference extension rule R is said to be *separable* if $\forall x, y, z \in \Omega$, $(y/\mathcal{C}^{y \neq z})\ R(x/\mathcal{C}^{y \neq z})\ (z/\mathcal{C}^{y \neq z}) \Rightarrow y\ R(x)\ z$, where $\mathcal{C}^{y \neq z} = \{q \in \mathcal{C} : y^q \neq z^q\}$. Under a separable extension rule, only any two committees are compared on the sole basis of the candidates on which they disagree.

The set of all top-consistent and separable extension rules is denoted by $\mathcal{A}$.

Furthermore, we restrict our attention to *rich* subsets of $\mathcal{A}$, where richness is defined as follows. Let C be any non-zero integer, let σ be a permutation of $\{1, ..., C\}$, let $x \in \{0,1\}^C$. Then we denote by $x_\sigma = (x_\sigma^1, ..., x_\sigma^C)$ the committee defined by: $\forall c$, $x_\sigma^c = 1 \Leftrightarrow x^{\sigma(c)} = 1$. Now let $R \in \mathcal{A}$; the extension rule $R^{x,\sigma}$ is defined by:

- $\forall C' \neq C$, $\forall y \in \{0,1\}^{C'}$, $R(y) = R^{x,\sigma}(y)$
- $\forall y, z \in \{0,1\}^C$, $[y\ R(x)\ z \Leftrightarrow y_\sigma\ R^{x,\sigma}(x_\sigma)\ z_\sigma]$

Then $\mathcal{B} \subset \mathcal{A}$ is *rich* if $\forall C$, $\forall \sigma$, $\forall x \in \{0,1\}^C$, $R \in \mathcal{B} \Leftrightarrow R^{x,\sigma} \in \mathcal{B}$.

Richness is not a very demanding property. Suppose that voters 1 and 2 respectively cast the ballots $(1,1,0)$ and $(0,1,1)$; moreover, suppose that we assume that $(1,0,0)\ R_1\ (0,1,0)$; this means that, in case where only one of the two approved candidates would be elected, voter 1 would prefer candidate 1 to be chosen. Based on the available information, how could one preclude that $(0,1,0)\ R_2\ (0,0,1)$? In other words, dropping $R_1^{x_1,\sigma}$ from the set of possible preferences, where $\sigma(1)=2$, $\sigma(2)=3$ and $\sigma(3)=1$, would imply more information that the actually available one. This does not happen if the set of possible rules is rich.

Once equipped with preferences over committees, we can define the Pareto efficiency of a voting rule. The voting rule V is said to be *Pareto-efficient* for $\mathcal{B} \subseteq \mathcal{E}$ if is always produces a Pareto-optimal committee (formally, if for any ballot set $X^{NC} \in \mathcal{X}$, for any vector $\rho = (R_1, ..., R_N) \in \mathcal{B}^N$, there is no $x \in \{0,1\}^C$ such that $x\ R_n\ V(X^{NC})$ for all $n \in \mathcal{N}$ and $x\ P_m\ V(X^{NC})$ for at least one $m \in \mathcal{N}$).

We address the following question:

Characterize the largest rich subset $\mathcal{E}^*$of $\mathcal{E}$ such that the issue-wise majority rule is Pareto-efficient for $\mathcal{E}^*$

Both top-consistency and separability are properties that obviously favor the Pareto-efficiency of all α-voting rules. Indeed, no voting rule can secure a Pareto outcome under a preference extension rule that fails to be top-consistent (consider the case where voters cast a unanimous ballot not describing their first-best committee). Moreover, any α-voting rule describes a separable choice, so that it cannot take into account any spillover effect that would prevail in the preferences over committees. As a result, it may be the case that issue-wise majority voting may select the unanimously least preferred committee[1].

2.3 Pareto-efficient voting: fixed size committees

The above definitions have to be adapted to the case where the committee to be chosen involves exactly k candidates. Preferences are now defined on Ω_k. Let Γ^{Ck} stand the set of all complete preorders on Ω_{Ck}, and let $\mathcal{R}^k = \cup_{Q \geq 1} \Gamma^{Ck}$.

A *preference extension rule* is a function $E = (R, k)$ defined from from $\Omega \times \mathbb{N}$ to R, where R is a preference extension rule, and $\mathbb{N}$ is the set of non-zero-integers, which associates with each ballot $x \in \Omega_C$ and each integer $0 < k < C$, an element $E(x,k)$ of Γ^{Ck} (with asymmetric part P and symmetric part I). A vector $\rho = (E_1, ..., E_N)$ of preference extension rules maps each ballot set X^{NC} and an integer k to a preference profile $\rho(X^{NC}) = (E_1(x_1,k), ..., E_N(x_N,k)) = (E_1^k, ..., E_N^k)$. The asymmetric part of E_n^k is denoted by P_n^k.

Suppose that $x \in \Omega_k$, that is the ballot fulfills the size requirement. Then top-consistency is defined as for the unrestricted case, provided that Ω replaced with Ω_k. But, as it is the case for the sequential approval rule, a ballot may contains more than k approvals. The following definition extends top-consistency to such a case:

Definition 4 *An extension rule E is top-consistent if:* $\forall k \in \mathbb{N}$, $\forall x \in \cup_{C \geq k} \{0,1\}^C$,
i. $[\mid 1(x) \mid \leq k \Rightarrow y\ P(x)\ z]$ *for all* $y, z \in \Omega_k$ *with* $1(x) \subseteq 1(y)$ *and* $1(x) \nsubseteq 1(z)$
ii. $[\mid 1(x) \mid > k \Rightarrow y\ P(x)\ z]$ *for all* $y, z \in \Omega_k$ *with* $1(y) \subseteq 1(x)$ *and* $1(z) \nsubseteq 1(x)$

Top-consistency is satisfied if, when a voter approves at most k candidates, she prefers any committee that includes those candidates than any other one, and if, when she approves more than k candidates, she would prefer any committee all members of which she approves than any other one. Note that the most preferred committee is always unique only if a voter approves more candidates than the number of eligible ones: if n casts ballot $x_n = (1,1,1,0)$ and $k = 2$, then $x = (1,1,0,0)$, $y = (1,0,1,0)$ and $z = (0,1,1,0)$ are strictly preferred to all other committees, and n might be indifferent between them.

[1] The reader may refer to Benoit and Kornhauser (1999), Lacy and Niou (2000), Ratliff (2003), (2006) for further analysis of non-separable preferences.

An immediate adaptation of the separability property works as follows: the preference extension rule E is said to be *separable* if $\forall k < k' \in \mathbb{N}$, $\forall x \in \cup_{C \geq k,k'} \{0,1\}^C$, $\forall\ y, z \in \Omega_k$, $[y\ E(x,k)\ z \Rightarrow (y,w)\ E(x,k')\ (z,w)]$, where $w \in \Omega_{k'-k}$ is such that $1(w) = k' - k$, and (y,w) stands for the committee y completed by all members in w. Separability is to be interpreted as follows: the way to compare two committees of a given size is not modified when both involve some common additional members. Note that separability does not impose any restriction on the ranking of fixed size committees, since it only relates to situations where this size varies[2].

The set of all top-consistent, neutral and separable extension rules is denoted by $\mathcal{E}^F$. The richness property is defined as above. Finally, the sequential approval rule S is *Pareto-efficient* for $\mathcal{B} \subseteq \mathcal{E}^F$ if for any k, any ballot set $X^{NC} \in \mathcal{X}$ such that $C \geq k$, any $\rho = (E_1, ..., E_N) \in \mathcal{B}^N$, any committee in $S_k(X^{NC})$ is Pareto-optimal in restriction to Ω_k, that is there is no $x \in \Omega_k$ such that $x\ E_n^k\ S_k(X^{NC})$ for all $n \in \mathcal{N}$ and $x\ P_m^k\ S_k(X^{NC})$ for at least one $m \in \mathcal{N}$.

We then address the next question:

Characterize the largest rich subset $\mathcal{E}^{**}$of $\mathcal{E}^F$ such that the sequential approval voting rule S is Pareto-efficient for $\mathcal{E}^{**}$

3 Results

3.1 Variable committee size

We first address the first question. It is shown in Özkal-Sanver and Sanver (2006) that any anonymous voting rule, hence in particular the issue-wise majority rule, is Pareto inefficient when any extension rule in $\mathcal{E}$ is allowed, As an illustration, consider the following example.

Example 1 Let $N = C = 3$, and consider the ballot set $X =$

c	1	2	3
x_1	1	1	0
x_2	0	1	1
x_3	1	0	1

. Let $\rho(X) = (R_1, R_2, R_3)$ be the preference profile on committees defined by[3]:

R_1	110	010	100	000	111	011	101	001
R_2	011	010	001	000	111	110	101	100
R_3	101	100	001	000	111	110	011	010

It is easily checked that $\rho \in \mathcal{E}^3$. Moreover, the issue-wise majority committee is $M(X) = (1,1,1)$, whereas all the voters prefer $(0,0,0)$ to $(1,1,1)$.

It follows that $\mathcal{E}^*$ is a proper subset of $\mathcal{E}$. We next argue that $\mathcal{E}^*$ is non-empty. Indeed, it contains the Hamming extension rule, which is defined as follows. Let $x = (x^1, ..., x^C)$, $y = (y^1, ..., y^C) \in \{0,1\}^C$ be any two ballots ; the *Hamming distance* between x and y is defined as $d(x,y) =\mid \{q = 1, ..., C : x^c \neq y^c\} \mid$. The *Hamming extension rule* R^{Ham} is then defined by: $\forall C$, $\forall x, y, z \in \{0,1\}^C$, $d(x,y) < d(x,z) \Leftrightarrow y$

[2] Separability for fixed size committees would work as follows: an extension rule E is *separable* if $\forall k, \forall\ C > k$, $\forall x \in \Omega_{Ck}$, $\forall y, z \in \Omega_k$ such that $\mid \mathcal{C}^{y \neq z} \mid = a\ [yR(x)z \Leftrightarrow (y/\mathcal{C}^{y\neq z}, w)R(x)(z/\mathcal{C}^{y\neq z}, w)]$, for all $w \in \{0,1\}^C$ such that $(y/\mathcal{C}^{y\neq z}, w)$ and $(z/\mathcal{C}^{y\neq z}, w) \in \Omega_k$.

[3] Committees are ranked from left to right according to the decreasing preference preorder, with the convention that all elements of some indifference class appear in a common cell.

$P^{Ham}_{(x)}$ z and $d(x,y) = d(x,z) \Leftrightarrow y\ I^{Ham}_{(x)}\ z$. The Hamming distance between two committees is the number of candidates they differ about. The Hamming extension rule then ranks committees according to their respective distance to the ballot. This clearly implies that all candidates are given an equal weight relative to their status: the same loss in satisfaction is obtained either when an approved candidate is not elected, or when a disapproved candidate is elected. Moreover, R^{Ham} is obviously both top-consistent and separable, and $\{R^{Ham}\}$ is rich.

It is easy to prove that for any ballot set X^{NC}, the issue-wise majority committee $M(X^{NC})$ minimizes on $\{0,1\}^C$ the total distance $\Sigma_{n=1}^{N} d(y, x_n)$ (see Brams et al., 2004, for a formal proof). Thus, the issue-wise majority rule is Pareto-efficient for $\{R^{Ham}\}$, so that $\mathcal{E}^*$ is non-empty. In fact, $\mathcal{E}^*$ contains all extension rule that can be built from R^{Ham} by allowing any way to cut ties within each indifference class. Say that a preference extension rule R is *Hamming consistent* if $\forall C, \forall x,y,z \in \{0,1\}^Q$, $d(x,y) < d(x,z) \Rightarrow y$ $P^{HamC}(x)\ z$. Then, we have the following proposition:

Proposition 1 *The issue-wise majority rule is Pareto-efficient for the set of Hamming consistent extension rules.*

Proof Let X^{NC} be any ballot set and $\rho = (R_1, ..., R_N)$ be a vector of Hamming consistent extension rules. Suppose that $y \in \{0,1\}^C$ Pareto dominates $z = M(X^{NC})$. Hence, $y\ R_n\ z$ for all n and $y\ P_{n^*}\ z$ for at least one n^*. Hamming-consistency implies that $d(x_n, y) \leq d(x_n, z)$ for all n, and thus $\Sigma_{n\in N} d(x_n, y) \leq \Sigma_{n\in N} d(x_n, z)$. Furthermore, since z minimizes $\Sigma_{n=1}^{N} d(x, x_n)$ on $\{0,1\}^C$, then $\Sigma_{n\in N} d(x_n, y) = \Sigma_{n\in N} d(x_n, z)$, which ensures that $d(x_n, y) = d(x_n, z)$ for all n. We may assume w.l.g. that $z^c = 0$ for all c (through a relevant relabelling of voters' positions regarding candidates).

Furthermore, note that $C' =\mid 1(y) \mid$must be an even integer, since in each R_n, both y and z belong to the same indifference class for R^{Ham}_n. Then, for all $c \in 1(y), (z/1(y))^q = 0$, and for all n, $\mid c \in 1(y) : (x_n/1(y))^c = 0 \mid = \mid c \in 1(y) : (x_n/1(y))^c = 1 \mid = C'/2$. It follows that $\Sigma_{n\in\mathcal{N}} \Sigma_{c\in 1(y)} (x_n/1(y))^c = NC'/2$ **(1)**

Finally, the definition of z implies that $\mid n \in \mathcal{N} : (x_n/1(y))^c = 1 \mid < N/2$ for all $c \in 1(y)$. Hence, for all $c \in 1(y)$, $\Sigma_{n\in\mathcal{N}} (x_n/B)^c < N/2 \Longrightarrow \Sigma_{c\in 1(y)} \Sigma_{n\in\mathcal{N}} (x_n/1(y))^c < C'N/2$, which contradicts **(1)** ♦

Note that Hamming consistency implies top-consistency, but allows for non-separability. Imposing separability drastically reduces the number of extension rules. For instance, in the 3-candidate case, 36 Hamming-consistent complete preorders can be built from a ballot, among which only 6 are separable.

In fact, the Pareto efficiency of issue-wise majority voting still hold for the following weakening of Hamming consistency. A preference extension rule R is *weakly Hamming-consistent* if $\forall C$, $\forall \alpha < \frac{C}{2}$, $\forall x, y \in \{0,1\}^C$, $d(x,y) \leq \alpha \Rightarrow y\ P(x)\ (-y)$,where $(-y)$ denotes the opposite committee of y, that is the committee such that: $\forall c$, $(-y)^c \neq y^c$.

The weak Hamming-consistency property only relates to the comparison of opposite committees: if a committee coincides with the cast ballot on more than half of the candidates, than this committee should be ranked higher than its opposite. Replacing Hamming consistency with weak Hamming consistency considerably enlarges the set of admissible extension rules: for instance, in the 4-candidate case, it allows for building 2^{24} more separable and top-consistent preferences from a ballot.

We denote by $\mathcal{E}^W$ the subset of $\mathcal{E}$ of all top-consistent, separable and weakly Hamming consistent extension rules. The next proposition shows that it is the larger rich subset of $\mathcal{E}$ which ensures that issue-wise majority voting is Pareto-efficient.

Proposition 2 $\mathcal{E}^* = \mathcal{E}^W$.

Proof We first prove that $\mathcal{E}^W \subseteq \mathcal{E}^*$. Let $\rho = (R_1, ..., R_N) \in (\mathcal{E}^W)^N$, and suppose that there exists some ballot set X^{NC} such that $M(X^{NC}) = w$ is Pareto-dominated by $x^* = (x^{*1}, ..., x^{*Q})$ in the profile $R(X^{NC}) = (R_1, ..., R_N)$. First, we claim that $\exists n \in \mathcal{N}$ such that $d(x_n, w) < \frac{C}{2}$.

Indeed, it follows from the definition of w that $\sum_{n\in\mathcal{N}} \mid \{c = 1, ..., C : x_n^c = w^c\} \mid > \frac{NC}{2}(*)$. Suppose that $d(x_n, w) \geq \frac{Q}{2}$ for all n. Then, $\forall n$, $\mid \{c : x_n^c = w^c\} \mid \leq \frac{C}{2}$, which contradicts with (*). Thus, there exists n^* such that $\mid \{c : x_{n^*}^c = w^c\} \mid > \frac{Q}{2}$, which means that $d(x_{n^*}, w) < \frac{C}{2}$.

It follows that $x^* \neq (-w)$: since $d(x_{n^*}, w) < \frac{C}{2}$, then weak Hamming consistency ensures that $w \; P_{n^*} \; (-w)$.

Finally, let $B = \{c = 1, ...C : y^c \neq w^c\}$. Note that $\mid B \mid = C' < C$. It follows from construction that $x^*/B = (-w/B)$. Furthermore, the same argument as above implies that $d(x_{n^{**}}/B, w/B) \leq \frac{C'}{2}$ for some $n^{**} \in \mathcal{N}$. Thus, one get from weak Hamming-consistency that $w/B \; P_{n^{**}} \; x^*/B$. Then, from the separability of R, it follows that $w \; P_{n^{**}} \; x^*$, which contradicts that x^* Pareto dominates w.

Next, we prove that $\mathcal{E}^* \subseteq \mathcal{E}^W$.

Let R be top-consistent, separable, but not weakly Hamming-consistent. Thus, $\exists C$, $\exists \alpha \leq \frac{C}{2}$ such that $\exists x, y \in \{0,1\}^Q$ which verify $d(x, y) = \alpha$ and $(-y) \; R(x) \; y$. One may assume without loss of generality that $y^c = 1$ for all c. We claim that there exists a ballot set X^{NC} such that $M(X^{NC})$ is Pareto-dominated.

Let $N = \beta + 1$, where $\beta = \binom{C}{\alpha} = \frac{C(C-1)...(C-\alpha+1)}{2.3.4....\alpha(\alpha-1)}$. Let X^{NC} be defined by:

- $\cup_{1\leq n\leq\beta}\{x_n\} = \{x \in \{0,1\}^C : \mid 1(x) \mid = C - \alpha\}$
- $\forall n, n' \in \{1, ..., \beta\}$, $x_n \neq x_{n'}$
- $x_N^c = 0$ for all $c = 1, ..., C$

Consider any candidate c. The number $1(c)$ of approvals given to c is $1(c) = \binom{C-1}{\alpha} = \frac{(C-1)...(C-\alpha)}{2.3.4....\alpha(\alpha-1)} = \frac{(C-\alpha)}{C}.\beta$.

Suppose that C is even. Thus, $\alpha \leq \frac{C}{2} - 1 \Rightarrow \frac{(C-\alpha)}{C}.\beta > (\frac{1}{2} + \frac{1}{C}).\beta$. It follows that, if β is even, then $1(c) > \frac{\beta}{2} + 1 > \frac{\beta+1}{2}$, while if β is odd, then $1(c) > \frac{\beta+1}{2} + \frac{\beta}{C} > \frac{\beta+1}{2}$. Suppose that C is odd. Thus $\alpha \leq \frac{C-1}{2} \Rightarrow \frac{(C-\alpha)}{C}.\beta > (\frac{1}{2} + \frac{1}{2C}).\beta$. It follows that, if β is even, then $1(c) > \frac{\beta}{2} + 1 > \frac{\beta+1}{2}$, and, if β is odd, then $1(c) > \frac{\beta+1}{2} + \frac{\beta}{2C} > \frac{\beta+1}{2}$. Hence, $y = M(X^{NC})$

By construction of X^{NC}, there exists $n^* \in \mathcal{N}$ such that $x = x_{n^*}$. Thus, $(-y) \; R_{n^*} \; y$. From the richness property, one can assume that $(-y) \; R_n \; y$ for all $n = 1, ..., \beta$. Moreover, top-consistency implies that $x_N = (-y) \; P_N \; y$. Hence, $(-M(X^{NC}))$ Pareto-dominates $M(X^{NC})$, which completes the proof ♦

3.2 Fixed committee size

We now turn to the case where a fixed number k of candidates has to be elected by means of the sequential approval rule S_{Ck}. Under such a rule, voters may approve as many candidates as they wish, and their ballots are extended to a complete preorder over the subset Ω_k of committees. First, we introduce a measure of the deviation that prevails between a committee and a ballot.

Definition 5 *Let $k < C$ be two integers. Let $x \in \{0,1\}^C$ and let $y \in \Omega_{Ck}$. The deviation between x and y is the integer $D(y \rightarrow x) = \mid \{c = 1, ..., C : (x/1(y))^c = 0\} \mid$, where $1(y) = \{c : y^c = 1\}$.*

The deviation between a committee y and a ballot x is the number of members of y who are disapproved in x. Replacing the Hamming distance with the deviation measure allows for defining the following notions of deviation and deviation-consistent k-extension rules:

- the *deviation extension rule* is the extension rule E^D defined by $\forall k \in \mathbb{N}$, $\forall x \in \cup_{C\geq k}\{0,1\}^C$, $\forall y, z \in \Omega_k$, $D(y \to x) < D(z \to x) \Leftrightarrow y\ P(x)\ z$ and $D(y \to x) = D(z \to x) \Leftrightarrow y\ I(x)\ z$

- an extension rule E is *D-consistent* if $\forall k$, $\forall x \in \cup_{C\geq k}\{0,1\}^C$, $\forall y, z \in \Omega_k$, $D(y \to x) < D(z \to x) \Leftrightarrow y\ P(x)\ z$

We denote by $\mathcal{E}^D$ the subset of all D-consistent extension rules.

Proposition 3 *Let $\mathcal{B}$ be a rich subset of $\mathcal{E}^D$. Then S is Pareto-efficient for $\mathcal{B}$ if and only if $\mathcal{B} = \{E^D\}$.*

Proof Sufficiency Part:

Let X^{NC} be a ballot set where $C \geq k$. It is obvious to check that $\{E^D\}$ is rich. Let $\rho = (E^D, ..., E^D)$. Let $x \in S_k(X^{NC})$ and suppose x is Pareto-dominated by $x^* \in \Omega_k$ in $E(X^{NC}) = (E_1, ..., E_N)$. It follows from definition that $\forall c \in x$, $\forall c' \notin x$, $\sum_{n\in\mathcal{N}} x_n^c \geq \sum_{n\in\mathcal{N}} x_n^{c'}$.

Since, x^* Pareto-dominates x, then it follows from the definition of E^D that, $\forall n$, $D(x^* \to x_n) \leq D(x \to x_n)$, and there exists n^* such that $D(x^* \to x_{n^*}) < D(x \to x_{n^*})$. This implies in turn that, $\forall n$, $\mid \{c : (x_n/1(x^*))^c = 1\} \mid \geq \mid \{c : (x_n/1(x))^c = 1\} \mid$ and $\mid \{c : (x_{n^*}/1(x^*))^c = 1\} \mid > \mid \{c : (x_{n^*}/1(x))^c = 1\} \mid$.

It follows that $\sum_{n\in\mathcal{N}} \sum_{c\in x^*} x_n^c > \sum_{n\in\mathcal{N}} \sum_{c\in x} x_n^c$, hence that $\sum_{c\in x^*} \sum_{n\in\mathcal{N}} x_n^c > \sum_{c\in x} \sum_{n\in\mathcal{N}} x_n^c$. This ensures the existence of $c^* \in (x^* - x)$ for whom $\sum_{n\in\mathcal{N}} x_n^{c^*} > \sum_{n\in\mathcal{N}} x_n^{c^*}$, which contradicts that $x \in S_k(X^{NC})$.

Necessary Part:

Let $E \neq E^D \in \mathcal{E}^D$. Then there exists $k \in \mathbb{N}$, $\exists x \in \cup_{C\geq k}\{0,1\}^C$, and $\exists y, z \in \Omega_k$, such that $D(y \to x) = D(z \to x) = a$ and $y\ P(x)\ z$.

Using separability, we can assume that $(y \cap z) = \varnothing$. The ballot x, together with the two committees y and z, are described in the following table:

c	1	...	a	$a+1$	...	k	$k+1$	...	$k+1+a$	$k+2+a$	...	$2k$	...
x	1	1	1	0	0	0	1	1	1	0	0	0	0
y	1	1	1	1	1	1	0	0	0	0	0	0	0
z	0	0	0	0	0	0	1	1	1	1	1	1	1

Let X^{NC}, where $N = 2.(k!)$ be defined as follows: let Δ_1 (resp. Δ_2) stand for the set of all permutations of $\{1, ..., k\}$ (resp. of $\{k+1, ..., 2k\}$). Then, for any pair of permutations $(\sigma, \mu) \in \Delta_1 \times \Delta_2$, there is a single voter n such that $x_n = ((x/\{1, ..., k\})_\sigma, (x/\{k+1, ..., 2k\})_\mu, (x/\{2k+1, ..., C\})$. It follows that there exists n^* such that $x_{n^*} = x$. Thus, $y\ P_{n^*}\ z$. furthermore, one can assume from the richness of $\mathcal{B}$ that $\forall n \in \mathcal{N}$, $y\ P_n\ z$, so that y Pareto dominates z. Finally, it is obviously checked that every candidate receives the same number of approvals, so that $z \in S_k(X^{NC})$ ♦

The proof of the necessary part rests upon the fact that, when the extension rule fails to be D-consistent, the sequential approval rule may allow for two committees or more, one of them being Pareto dominated by another. In fact, the D-consistency property is necessary and sufficient for the Pareto efficiency of sequential approval voting, under the further restriction that ballot sets always produce a unique outcome:

Proposition 4 *Let $\mathcal{X}^1$ denote the set of all ballot sets such that $\forall k$, $\forall C > k$, $\forall X^{NC} \in \mathcal{X}^1$, $\mid S_{Ck}(X^{NC}) \mid = 1$. Let $\mathcal{B} \subseteq \mathcal{E}^F$ be a rich subset. Then S is Pareto efficient for $\mathcal{B}$ if and only if $\mathcal{B} = \mathcal{E}^D$.*

Proof Sufficiency Part:

Let $\rho = (E_1, ..., E_N) \in (\mathcal{E}^D\ )^N$. Let X^{NC} be a ballot set where $C \geq k$, and $\{x\} = S_k(X^{NC})$. Thus, $\forall c \in x$, $\forall c' \notin x$, $\sum_{n\in\mathcal{N}} x_n^c \geq \sum_{n\in\mathcal{N}} x_n^{c'}$.

Suppose that there exists $x^* \in \Omega_k$ such that: $x^* E_n\ x$, for all voters n, and there exists at least one n^* for whom $x^* P_{n^*}\ x$. It follows from D-consistency that $\forall n$, $D(x^* \to x_n) \leq D(x \to x_n)$. Hence, one get that,

$\forall n$, $\mid \{c : (x_n/1(x^*))^c = 1\} \mid \geq \mid \{c : (x_n/1(x))^c = 1\} \mid$. It follows that $\sum_{n\in\mathcal{N}} \sum_{c\in x^*} x_n^c \geq \sum_{n\in\mathcal{N}} \sum_{c\in x} x_n^c$, hence that $\sum_{c\in x^*} \sum_{n\in\mathcal{N}} x_n^c \geq \sum_{c\in x} \sum_{n\in\mathcal{N}} x_n^c$. Since $\forall c \in x$, $\forall c' \notin x$, $\sum_{n\in\mathcal{N}} x_n^c \geq \sum_{n\in\mathcal{N}} x_n^{c'}$, then $\sum_{c\in x^*} \sum_{n\in\mathcal{N}} x_n^c = \sum_{c\in x} \sum_{n\in\mathcal{N}} x_n^c$. This implies that $\forall c \in x, \forall c' \in x^*$, $\sum_{n\in\mathcal{N}} x_n^c = \sum_{n\in\mathcal{N}} x_n^{c'}$. Thus, $x^* \in S_{Ck}(X^{NC})$, which contradicts that $S_k(X^{NC})$ is unique.

Necessary part:

We use the same construction as in the precedent proof. Let $E \in (\mathcal{E}^F - \mathcal{E}^D)$. Hence, there exists $k \in \mathbb{N}$, $\exists x \in \cup_{C\geq k}\{0,1\}^C$, $\exists y, z \in \Omega_k$, such that $D(y \to x) = a < D(z \to x) = b$ and $z\ P(x)\ y$. From separability, we can assume that $(y \cap z) = \varnothing$, where $y^c = 1 \Leftrightarrow 1 \leq c \leq k$, $z^c = 1 \Leftrightarrow k+1 \leq c \leq 2k$, $x^c = 1 \Leftrightarrow$ either $1 \leq c \leq a$, or $k+1 \leq c \leq k+b$. Let X^{NC} be the ballot set defined as in the above proof. By the richness property, one get that $\forall n \in \mathcal{N}$, $z\ P_n\ y$, so that z Pareto dominates z.

Moreover, the symmetry of X^{NC} ensures that:

- $\forall c, h \in y$, $\sum_{n\in\mathcal{N}} x_n^c = \sum_{n\in\mathcal{N}} x_n^h = \binom{k}{a-1}$
- $\forall c', h' \in z$, $\sum_{n\in\mathcal{N}} x_n^{c'} = \sum_{n\in\mathcal{N}} x_n^{h'} = \binom{k}{b-1}$

Finally, since $a > b$, then $\forall c \in y$, $\forall c' \in z$, $\sum_{n\in\mathcal{N}} x_n^c > \sum_{n\in\mathcal{N}} x_n^{c'}$, and thus $\{y\} \in S_k(X^{NC})$ ♦

The proof above brings two by-products. First, under D-consistency, an outcome of the sequential approval rule can only be Pareto-dominated by another outcome of the rule ; second, the largest rich subset of $\mathcal{E}^F$ for which sequential approval voting is Pareto efficient reduces to a unique rule, namely the deviation extension rule:

Proposition 5 *(1) Let $\mathcal{B} \subseteq \mathcal{E}^D$ being such that S is not Pareto-efficient for $\mathcal{B}$. Then $\forall k$, $\forall C \geq k$, if $x \in S_k(X^{NC})$ is Pareto dominated by $y \in \Omega_k$ for a ballot set X^{NC}, then $y \in S_k(X^{NC})$.*

(2) $\mathcal{E}^{**} = \{E^D\}$

Proof Assertion (1) immediately follows from the proof of the sufficiency part in the precedent Proposition 4. The necessary Part in the same proof ensures that $\mathcal{E}^{**} \subseteq \mathcal{E}^D$. Assertion (2) then follows from Proposition 3 ♦

4 Further comments

4.1 Pareto ε-efficient voting

We now address the following question: do results similar to the previous ones hold if Pareto efficiency is replaced with a weaker property? Say that the voting rule V is said to be *Pareto ε-efficient* for $\mathcal{B} \subseteq \mathcal{E}$ if, for any of its outcome, no coalition containing at least ε% of the voters can form and make all of its members strictly better off by imposing another committee (formally, if $\forall\ X^{NC} \in \mathcal{X}$, $\forall \rho = (R_1, ..., R_N) \in (\mathcal{B})^N$, $\forall x \in \{0,1\}^C$,$\mid \{n \in \mathcal{N} : x\ P_n\ V(X^{NC})\ \} \mid < \varepsilon.N$). Weak Pareto efficiency is equivalent to 1-efficiency[4]. We prove below that issue-wise majority voting is not Pareto ε-efficient for the Hamming extension rule, even is ε is chosen as close as wanted to 1.

Proposition 6 *For any $\varepsilon' \in]0,1[$, there exists $\varepsilon > (1-\varepsilon')$ such that issue-wise majority voting is not Pareto ε-efficient for $\{R^{Ham}\}$.*

[4] The case where $\varepsilon = 0.5$ relates to the well-known Ostrogorski paradox (the reader may refer to Laffond and Laine (2006), (2009) and the references quoted there for further discussion).

Proof Let $C = 2.Q + 1$, where $Q > 1$. Consider the following set $\mathcal{N}_Q = \{1, ..., N_Q\}$ of voters and the ballot set $X^{N_Q C}$: $\mathcal{N}_Q = \mathcal{N}_Q^1 \cup \mathcal{N}_Q^2$ where,

- $\forall n \in \mathcal{N}_Q^1$, $1(x_n) = Q + 1$
- $\forall n \neq m \in \mathcal{N}_Q^2$, $x_n \neq x_m$ ($\Rightarrow \mid \mathcal{N}_Q^1 \mid = \begin{pmatrix} 2.Q+1 \\ Q+1 \end{pmatrix}$)
- $\forall n \in \mathcal{N}_Q^2$, $1(x_n) = 0$

Let us compute $M(X^{N_Q C}) \in \{0,1\}^{2.Q+1}$. The number $1(\mathcal{N}_Q^1, c)$ (resp. $0(\mathcal{N}_Q^1, c)$) of approvals (resp. disapprovals) received by a candidate c from voters in $\mathcal{N}_Q^1$ is equal to $\begin{pmatrix} 2.Q \\ Q \end{pmatrix}$ (resp. $\begin{pmatrix} 2.Q \\ Q+1 \end{pmatrix}$. It follows that $1(\mathcal{N}_Q^1, c) - 0(\mathcal{N}_Q^1, c) = \frac{(2.Q).(2.Q-1)...(Q+2)}{(Q).(Q-1)...(3).(2)} = \frac{1}{Q}.\begin{pmatrix} 2.Q \\ Q+1 \end{pmatrix}$.

Let $N_Q^2 = \mid \mathcal{N}_Q^2 \mid = E[\frac{1}{Q}.\begin{pmatrix} 2.Q \\ Q+1 \end{pmatrix}]$, that is the smallest integer strictly greater than $1(\mathcal{N}_Q^1, c) - 0(\mathcal{N}_Q^1, c)$. This ensures that $M(X^{N_Q C}) = (0, 0, ..., 0)$. Thus, it follows from the definition of R^{Ham} that, for every voter $n \in \mathcal{N}_Q^1$, $(-M(X^{N_Q C}))\ P_n^{Ham}\ M(X^{N_Q C})$. Finally, it is easy to check that $\frac{N_Q}{N_Q^1} \sim 1 + \frac{[(2Q).(.Q-1)...(Q+2)/Q.(Q-1)...(3).(2)]}{[(2Q+1).(2Q)...(Q+2)/(Q).(Q-1)...(2)]} = 1 + \frac{1}{2Q+1}$. Hence, $Lim_{Q\to\infty}[\frac{N_Q}{N_Q^1}] = 1$, which ensures the result ♦

Pareto ε-efficiency can also be defined in the case of fixed committee size. The sequential approval rule S is said to be *Pareto ε-efficient* for the extension rule $\mathcal{B} \subseteq \mathcal{E}^F$ if $\forall k$, $\forall C \geq k$, $\forall\ X^{NC} \in \mathcal{X}$, $\forall \rho = (E_1, ..., E_N) \in (\mathcal{B})^N$, $\forall x \in \Omega_k$,$\mid \{n \in \mathcal{N} : x\ P_n^k\ S_k(X^{NC})\ \} \mid < \varepsilon.N)$.

Proposition 7 *For any $\varepsilon \in]0,1[$, there is no preference extension rule $E \in \mathcal{E}^F$ for which the sequential approval rule is Pareto ε-efficient for E.*

Proof To be inserted here ♦

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