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Pairwise kidney exchange with age based preferences

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Abstract

We consider a model of Kidney-Exchange where compatible kidneys have different quality. Namely, a kidney from a young donor has higher quality than a kidney from an older one. We assume that the central planner does not observe which is the set of compatible kidneys for each patient, but it knows the age of all potential donors. We show that age based priority rules are group strategy-proof, k-efficient and individually rational. We also show how to deal with the case of multiple donors.

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Pairwise Kidney Exchanges with Age Based Preferences*

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Abstract

We propose a model of Kidney-Exchange that incorporates the main European institutional features. We assume that patients do not consider all compatible kidneys homogeneous and patients are endowed with reservation values over the minimal quality of the kidney they may receive. Under feasibility constraints, patients'

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truthful revelation of reservation values is incompatible with constrained efficiency. In light of this result, we introduce an alternative behavioral assumption regarding patients' incentives. Patients choose their revelation strategies to "protect" themselves from bad outcomes and use a lexicographic refinement of maximin strategies. In this environment, if exchanges are pairwise, then priority rules or rules that maximize a fixed ordering provide incentives for patients to report their true reservation values. The positive result vanishes if larger exchanges are admitted.

1 Introduction

2 Kidney Assignment Problems

Consider a finite society consisting of a set $N = \{1, \ldots, n\}$ of patients ($n \geq 3$) who need a kidney for transplantation. Each patient has a potential donor, and $\Omega = \{\omega_1, \ldots, \omega_n\}$ denotes the set of kidneys available for transplantation. For each patient i ω_i refers to the kidney of patient i's donor. We assume for the moment that each patient has only one potential donor and that there are not kidneys without living donor.

An ***assignment*** a is an n-tuple of pairs $a = [(1, \omega), \ldots, (n, \omega')]$ such that i) for each $i, j \in N$, $i \neq j$ and each $\omega, \omega' \in \Omega$, if $(i, \omega), (j, \omega') \in a$, then $\omega \neq \omega'$. Let $\mathcal{A}$ be the set of all assignments.

An assignment is an allocation of the available kidneys to the patients. For each patient i and each assignment a, we denote by a_i the kidney that patient i receives at a. If a patient does not receive a kidney from other patient donors', then she continues in dyalisis, and we write that she is assigned her own initial donor. Of course, an assignment never allocates the same kidney to two different patients.

Each patient i is equipped with a complete and transitive preference relation $\succsim_i$ on Ω. We denote by $\succ_i$ the associated strict preference relation. While we do not rule out indifferences among different kidneys, we assume that for each $\omega \neq \omega_i$ either $\omega \succ_i \omega_i$

or $\omega_i \succ_i \omega$. For each patient i let $\mathcal{P}_i$ denote the set of all her admissible preferences. For each patient i and each preference $\succsim_i \in \mathcal{P}_i$, define her sets of compatible kidneys $C(\succsim_i) \equiv \{\omega \in \Omega \mid \omega \succ_i \omega_i\}$ and incompatible kidneys $I(\succsim_i) \equiv \{\omega \in \Omega \mid \omega_i \succ_i \omega\}$.

In every assignment, kidneys are allocated by forming exchange cycles of patient–donors couples. In each cycle, every patient receives a kidney from the donor of some patient in the cycle and simultaneously her donor's kidney is transplanted to another patient in the cycle. In an exchange cycle among k couples, all the kidneys must be reaped from the donors and transplanted to the patients simultaneously.

For each assignment a, let π_a be the finest partition of the set of patients such that for each $p \in \pi_a$ and each $i \in p$:

1. [(i)]
2. either there are $j, j' \in p$, with $a_i = \omega_j$ and $a_{j'} = \omega_i$,[1]
3. or $a_i = \omega_i$.

Clearly, for each assignment a the partition π_a is unique and well-defined. We define the ***cardinality of*** a as the $max_{p \in \pi_a} \# p$.

The cardinality of an assignment refers to the size of the largest cycle formed in the assignment. Basically, it refers to the maximum number of simultaneous operations involved in an assignment. Of course, the concept of cardinality is crucial for our notion of feasibility.

For each $k \in \mathbb{N}$, $k \leq n$, we say that the assignment a is k–***feasible*** if a's cardinality is not larger than k. Let $\mathcal{A}^k$ be the set of all k–feasible assignments.

An interesting case of feasibility restrictions appears when only immediate exchanges between two couples are admitted. An assignment a is a *pairwise-exchange* assignment ($a \in \mathcal{A}^2$) if a satisfies that if for some $i, j \in N$ $(i, \omega_j) \in a$, then $(j, \omega_i) \in a$.

[1]Note that $j = j'$ and $i = j = j'$ and then $a_i = \omega_i$ are allowed.

3 Kidney Assignment Rules

In this paper, we are interested in rules that select a (kidney assignment) for each (kidney exchange) problem. An ***(assignment) rule*** is a mapping φ that selects an assignment a for each preference profile.

Let $\mathcal{D} \equiv \times_{i\in N} D_i \subseteq \times_{i\in N} \mathcal{P}_i$ be a domain of preferences over kidneys. A rule is a mapping $\varphi : \mathcal{D} \to \mathcal{A}$. For each patient i and each preference profile $\succsim$, we denote by $\varphi_i(\succsim)$ the object assigned to i by φ.

Next, we present formal definition of the standard conditions for desirable rules.

Individual Rationality. For each $i \in N$ and each $\succsim \in \mathcal{D}$, $\varphi_i(\succsim) \succsim_i \omega_i$.

k−***Efficiency.*** For each $\succsim \in \mathcal{D}$, $\varphi(\succsim) \in \mathcal{A}^k$ and there is no $a \in \mathcal{A}^k$ such that for each $i \in N$ $a_i \succsim_i \varphi_i(\succsim)$ and for some $j \in N$, $a_j \succsim_j \varphi_j(\succsim)$.

Strategy–proofness. For each $i \in N$, each $\succsim \in \mathcal{D}$, and each $\succsim'_i \in D_i$,

$$\varphi_i(\succsim) \succsim_i \varphi_i(\succsim'_i, \succsim_{-i}).$$

4 Incentive-Compatibility and Feasibility Constraints

We say that a preference domain $\mathcal{D} \equiv \times_{i\in N} D_i$ is rich if for each patient i and each $\omega, \omega' \in \Omega \setminus \{\omega_i\}$, there are $\succsim_i, \succsim'_i \in D_i$ such that

1. [(i)]
2. $\omega \succ_i \omega' \succ_i \omega_i \succ_i \omega''$ for each $\omega'' \notin \{\omega, \omega', \omega_i\}$.
3. $\omega \succ'_i \omega_i \succ'_i \omega''$ for each $\omega'' \notin \{\omega, \omega_i\}$.

Theorem 1. *Let $2 \leq k \leq n-1$. No rule defined on a rich domain satisfies* individual rationality, *k*-efficiency*, and* strategy-proofness.

Proof. We study separately two cases. We analyze first the restriction to pairwise exchanges. Then, we provide the proof for $k \geq 3$. In both cases we exploit arguments similar to those employed in the literature of strategy-proof assignment rules in economies with indivisibilities where the core is empty ($k = 2$, roommate problem) or multi-valued ($k \geq 3$).[2]

Assume to the contrary there is a rule φ defined on a rich domain $\mathcal{D}$ that satisfies *individual rationality*, 2-*efficiency*, and *strategy-proofness.* Consider three patients $\{1, 2, 3\}$ and a preference profile $\succsim \in \mathcal{D}$ such that its restriction to these patients and their donors' kidneys is:

$\succsim_1$	$\succsim_2$	$\succsim_3$
ω_2	ω_3	ω_1
ω_3	ω_1	ω_2
ω_1	ω_2	ω_3

and so that for each $i \in \{1, 2, 3\}$, and each $j \notin \{1, 2, 3\}$, $\omega_j \succ_j \omega_i$ and $\omega_i \succ_i \omega_j$. By *individual rationality*, and in order to simplify notation, we can assume that $N = \{1, 2, 3\}$.

By *individual rationality* and 2-*efficiency*, φ selects an assignment in which two patients exchange their donors' kidneys while the remaining patient receives the null kidney. We assume without loss of generality that $\varphi(\succsim) = [(1, \omega_2), (2, \omega_1), (3, \omega_3)]$.

Next, let $\succsim'_1$ be such $\omega_2 \succ'_1 \omega_1 \succ'_1 \omega'$ for each $\omega' \notin \{\omega_1, \omega_2\}$. By *strategy–proofness*, $\varphi_1(\succsim'_1, \succsim_{-1}) = \omega_2$. Finally, let $\succsim'_2$ be such $\omega_3 \succ'_2 \omega_2 \succ'_2 \omega''$ for each $\omega'' \notin \{\omega_2, \omega_3\}$. By *individual rationality* and *strategy-proofness*, $\varphi_2(\succsim'_{-3}, \succsim_3) = \omega_2$. Then, $\varphi(\succsim'_{-3}, \succsim_3) = [(1, \omega_1), (2, \omega_2), (3, \omega_3)]$. Note that the assignment $a = [(1, \omega_1), (2, \omega_3), (3, \omega_2)]$ is 2–*feasible*, and $a_i \succsim_i \varphi_i(\succsim'_{-3}, \succsim_3)$ for each $i \in N$ and $a_2 \succ'_2 \varphi_2(\succsim'_{-3}, \succsim_3)$. Then, φ violates 2–*efficiency.*

Next, we analyze the general case. Let $k \geq 3$. Remember that $k < n$ and then there are at least $k+1$ patients. Assume to the contrary there is a rule φ that satisfies *individual*

[2] See **?**.

rationality, *k-efficiency*, and *strategy-proofness*. Let the preference profile $\succsim \in \mathcal{D}$ be such that for each $i = 1 \ldots, k$

$$\omega_{i+1} \succ_i \omega_i \succ_i \omega \quad \forall\, \omega \in \Omega \setminus \{\omega_i, \omega_{i+1}\},$$

and for patient $k+1$

$$\omega_1 \succ_{k+1} \omega_2 \succ_{k+1} \omega_{k+1} \succ_{k+1} \omega \quad \forall\, \omega \in \Omega \setminus \{\omega_1, \omega_2, \omega_{k+1}\},$$

By *individual rationality*, we can simplify notation and assume that $N = \{1, \ldots, k+1\}$.

Note that, by *individual rationality* either no object is assigned to any patient $1, \ldots, k+1$, or patient $k+1$ receives ω_2, patient 1 receives the null object, and every other patient i receives ω_{i+1} (the kidney of her next to the right neighbor). By *k-efficiency*:

$$\varphi(\succsim) = \begin{bmatrix} (1, \omega_1), & \\ (i, \omega_{i+1}), & \forall i = 2, \ldots, k \\ (k+1, \omega_2) & \end{bmatrix}.$$

Let $\succsim' \in \mathcal{D}$ be such that $\succsim'_i = \succsim_i$ for each $i \neq k-1$ and

$$\omega_k \succ'_{k-1} \omega_{k+1} \succ'_{k-1} \omega_{k-1} \succ_{k-1} \omega' \quad \forall\, \omega \in \Omega \setminus \{\omega_{k-1}, \omega_k, \omega_{k+1}\}.$$

By *strategy-proofness*, $\varphi_{k-1}(\succsim') \succsim'_{k-1} \varphi_{k-1}(\succsim) = \omega_k$. Note that ω_k is patient $k-1$'s preferred kidney. Then, $\varphi_{k-1}(\succsim') = \omega_k$. By *k-efficiency* and *individual rationality*, $\varphi(\succsim') = \varphi(\succsim)$.

Let $\succsim'' \in \mathcal{D}$ be such that for each $i \neq k+1$, $\succsim''_i = \succsim'$ and

$$\omega_1 \succ''_{k+1} \omega_{k+1} \succ''_{k+1} \omega \quad \forall\, \omega \in \Omega \setminus \{\omega_1, \omega_{k+1}\}$$

The same arguments we employed to determine $\varphi(\succsim)$ apply to obtain:

$$\varphi(\succsim'') = \begin{bmatrix} (i, \omega_{i+1)}(\text{modulo } k+1), & \forall i \notin \{k, k-1\} \\ (k-1, \omega_{k+1}), & \\ (k, \omega_k) & \end{bmatrix}.$$

Note that $\omega_1 = \varphi_{k+1}(\succsim'') \succ'_{k+1} \varphi_{k+1}(\succsim') = \omega_2$, which contradicts *strategy-proofness*.

The previous impossibility result is robust to the introduction of weak preferences over kidneys. All we require is to admit the existence of two indifference classes for acceptable kidneys. Hence, Theorem 1 contrasts with the positive results in dichotomous domains of preferences by **?** and **?**.

5 Priority Rules and Age–Based Preferences

In the previous section we assume that patients' preferences are private information, and moreover they are completely independent across patients. Whereas, it is clear that many individual characteristic affect the compatibility between a patient and her potential donors, there are also observable features of the potential donors which affect in the same way to all patients, namely donor's age.

This fact induces structure in the domain of preferences of the patients that we can use to overcome the previous impossibility results.

From now on, we assume that we can partition the set of available kidneys in two disjoint subsets $\Omega \equiv Y \cup M$, young donors Y, and mature donors M. We assume $\#Y \geq 2$ $\#M \geq 2$.[3] This partition induces a natural restriction on patients' preferences. Whether being a young or a mature donor kidney does not determine the compatibility of the donor and the patient, a compatible young donor is preferred to a mature compatible donor. Moreover, patients are indifferent between any pair of young (mature) compatible donors.

For each patient i, we say that preference $\succsim_i \in \mathcal{P}_i$ is an age-based preference iff for each $\omega \in Y$ such that $\omega \succ_i \omega_i$,

1. [(i)]
2. $\omega \succsim_i \omega'$ for each $\omega' \in \Omega$, and

[3]While our results hold for the cases where $\#Y = 1$ or $\#M = 1$, they are ruled out in order to ease the exposition. These cases entail special relations that do not hold in general.

3. $\omega \succ_i \omega''$ for each $\omega'' \in M$.

Let $\mathcal{E}_i$ be the set of all patient i's age-based preferences.

For each $\succsim \in \mathcal{P}$, let $\mathcal{I}(\succsim) \equiv \{a \in \mathcal{A}^2 \mid a_i \succsim_i \omega_i\}$ denote the set of all individually rational pairwise assignments.

The previous literature on kidney exchange has focussed on priority mechanisms which are commonly used in most transplant centers to allocate cadaver organs. In this section, we analyze how priority mechanism need to be tailored to the age-based preferences environment

A priority ordering is a permutation of patients such that the k-th patient in the permutation is the patient with the k-th priority. Just for the sake of convenience, in the definition let the priority ordering of patients be the natural ordering.

A priority mechanism produces an assignment as follows. For each $\succsim \in \mathcal{P}$ and the priority ordering $(1, 2, \ldots, n)$ among the patients:

- Let $\mathcal{M}^0 = \mathcal{I}(\succsim)$.
- For each $k \leq n$, let $\mathcal{M}^k \subseteq \mathcal{M}^{k-1}$ be such that:

$$\mathcal{M}^k = \begin{cases} \{a \in \mathcal{M}^{k-1} : a_k \in Y\} & if\ \exists a \in \mathcal{M}^{k-1}\ s.t\ a_k \in Y, \\ \{a \in \mathcal{M}^{k-1} : a_k \in M\} & if \left(\begin{array}{c} \nexists a \in \mathcal{M}^{k-1}\ s.t\ a_k \in Y,\ and \\ \exists a \in \mathcal{M}^{k-1}\ s.t\ a_k \in M \end{array} \right); \\ \mathcal{M}^{k-1} & otherwise. \end{cases}$$

Group Strategy–proofness. For each $T \subseteq N$, each $\succsim \in \mathcal{D}$, and each $\succsim'_T \in D_T$,

$$\text{for each } i \in T \quad \varphi_i(\succsim) \succsim_i \varphi_i(\succsim'_T, \succsim_{N \setminus T}).$$

Lemma 1. *Let $\varphi : \mathcal{E} \to \mathcal{A}^2$ satisfy group*-strategy-proofness *and* individual rationality*, then if $\omega_i, \omega_j \in Y$ and $\omega_i \in C(\succsim_j)$ and $\omega_j \in C(\succsim_i)$ then either $\varphi_i(\succsim) \in Y \setminus \{\omega_i\}$ or $\varphi_j(\succsim) \in Y \setminus \{\omega_j\}$ (or both).*

Proof. Assume to the contrary that φ satisfies group-*strategy-proofness* and *individual rationality* there are i, j and $\succsim \in \mathcal{E}$ such that $\omega_i, \omega_j \in Y$ and $\omega_i \in C(\succsim_j)$ and $\omega_j \in C(\succsim_i)$, but $\varphi_i(\succsim) \notin Y$ or $\varphi_j(\succsim) \notin Y$. Let $\succsim'_{ij} \in \mathcal{E}$ be such that $\omega_j = C(\succsim'_i)$ and $\omega_i = C(\succsim'_j)$. By *individual rationality*, i and j cannot be matched to any other patient. By 2-*efficiency*, $\varphi_i(\succsim'_{ij}, \succsim_{-ij}) = \omega_j$ and $\varphi_j(\succsim'_{ij}, \succsim_{-ij}) = \omega_i$, which contradicts, group-*strategy-proofness.*

The same arguments are valid to prove the following parallel lemma.

Lemma 2. *Let $\varphi : \mathcal{E} \to \mathcal{A}^2$ satisfy group-*strategy-proofness *and* individual rationality*, then if $\omega_i \in Y$, $\omega_j \in M$ and $\omega_i \in C(\succsim_j)$ and $\omega_j \in C(\succsim_i)$ then either $\varphi_i(\succsim) \neq \{\omega_i\}$ or $\varphi_j(\succsim) \in Y$ (or both).*

The previous lemmata has important implications.

Example 1. *Let $N = \{1, 2, 3, 4\}$ and $Y = \{\omega_1, \omega_2\}$ $M = \{\omega_3, \omega_4\}$. Consider the preference profile $\succsim \in \mathcal{E}$ such that*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
ω_2	ω_1	ω_1	ω_2
$\omega_3 \sim_1 \omega_4$	$\omega_3 \sim_1 \omega_4$	ω_3	ω_4
ω_1	ω_2	$\dots$	$\dots$

Note that $[(1, \omega_3), (2, \omega_4), (3, \omega_1), (4, \omega_2)]$ is an individually rational assignment. However, every group-strategy-rule selects at this profile the assignment $[(1, \omega_2), (2, \omega_1), (3, \omega_3), (4, \omega_4)]$.

`We may introduce some comment on the maximizing the number of exchanges between young donors.`

Corollary 1. *A rule that maximizes the number of performed (individually rational) transplants violates group strategy-proofness.*

This corollary makes transparent the tension between information and an important normative requirement that most of transplant centers would like to be satisfied by the rules they use. This corollary also clarifies how the weak enlargment of the preference domain (by simply allowing patients to prefer young compatible donors than mature compatible one) induce drammatic difference with respect to the Roth et al,. dichotomous domain. It is important to stress that group-strategy-proofness seems a reasonable requirement (whenever we are interested in guaranteeing that patients have dominant strategies) because the possibility of coalition manipulation derives from the fact that a doctor or (a transplant center) has usually more than one patient and she may easily report manipulated information regarding more than one of her patients if by doing so she can increase the welfare of all of them. .

An age-based priority rule, is a priority mechanism such that for every i, j such that $\omega_i \in Y$ and $\omega_j \in M$, then $\pi(i) < \pi(j)$, where $\pi(i)$ denotes the priority of patient i in the priority mechanism.

Theorem 2.

1. *A 2-efficient priority rule satisfies* strategy-proofness *only if it is an age-based priority rule.*
2. *An age-based priority mechanism φ defined on $\mathcal{E}$ is group-strategy-proof and 2-efficient.*

Proof

1. Let $\omega_i \in Y$, $\omega_j \in M$. Let φ be a priority mechanism with priority ranking π such that $\pi(j) < \pi(i)$. Let $\omega_k \in M$ with $\pi(j) < \pi(k)$ (whose existence is guaranteed by the assumption that $\#M \geq 2$). Let $\succsim \in \mathcal{E}$ be such that $C(\succsim_i) = \{\omega_k\}$, $C(\succsim_j) = \{\omega_k\}$, and $C(\succsim_k) = \{\omega_i, \omega_j\}$. According to the priority rule, $\varphi_k(\succsim) = \omega_j$. Let $\succsim'_k \in \mathcal{E}_k$ be such that $C(\succsim'_k) = \{\omega_i\}$. Then, by 2-efficiency $\varphi_k(\succsim'_k, \succsim_{-k}) = \omega_i$, which violates *strategy-proofness.*
2. *Group Strategy-proofness.* Without loss of generality, let $\pi(i) = i$ for all $i \in N$ be the priority ordering of the rule φ. Consider agent 1. Suppose $\varphi_1(\succsim) \in M$ or $\varphi_1(\succsim) = \omega_1$ and there exists $T \ni 1$ and $\succeq, \succeq' \in \mathcal{E}$ such that $\varphi_1(\succsim'_T, \succsim_T) \succ_1 \varphi_1(\succsim)$.

Then $\varphi_1(\succsim'_T, \succsim_T) = \omega_j$ for some $j \neq 1$. By definition of the age-based priority rule for each $j \in N$ such that $\omega_1 \in C(\succsim_j)$, $\omega_1 \succeq_j \omega_k$ for all $k \in N$. Then, if $\varphi_1(\succsim) \neq \omega_j$ and $\varphi_1(\succsim'_T, \succsim_T) = \omega_j$ it follows that $\omega_1 \notin C(\succsim_j)$ which implies that the rule φ is not individually rational. Contraddiction. Suppose $\varphi_2(\succsim) \in M$ or $\varphi_2(\succsim) = \omega_2$ and there exists $T \ni 2$ and $\succeq, \succeq' \in \mathcal{E}$ such that $\varphi_2(\succsim'_T, \succsim_T) \succ_2 \varphi_2(\succsim)$. Then $\varphi_2(\succsim'_T, \succsim_T) = \omega_j$ for some $j \neq 2$.. Two cases must be considered: *i)* $\varphi_1(\succsim) = \omega_j$, then $\omega_1 \succeq_j \omega_2$ and therefore $j \notin T$. Moreover, it must be that $\varphi_1(\succsim'_T, \succsim_T) \prec_1 \omega_j$, which cannot hold since patient 1 has a priority with respect to patient 2. *ii)* $\varphi_1(\succsim) \neq \omega_j$; if $\omega_2 \in C(\succsim_j)$, then $\omega_2 \succeq_j \omega_k$ for all $k \in N \backslash \{1\}$; hence $\varphi_2(\succsim) \neq \omega_j$ implies $\omega_2 \notin C(\succsim_j)$. Consider now any agent $k \in N$. Suppose there exists $T \ni k$ such that $\varphi_k(\succsim'_T, \succsim_T) \succ_k \varphi_k(\succsim)$. Then $\varphi_k(\succsim'_T, \succsim_T) = \omega_j$ for some $j \neq k$. By induction for all $i < k$, $i \notin T$. Iterating the previous argument, two cases must be considered: *i)* $\varphi_i(\succsim) = \omega_j$ for some $i < k$, since $\omega_i \succeq_j \omega_k$ therefore $j \notin T$. Moreover there exists $l < k$ such that $\omega_j \in C(\succsim_j)$, $\varphi_l(\succsim'_T, \succsim_T) \prec \omega_j$, contraddicting that l has a priority over k; *ii)* $\varphi_i(\succsim) \neq \omega_j$ for all $i < k$; if $\omega_k \in C(\succsim_j)$, then $\omega_k \succeq_j \omega_m$ for all $m \in \{k+1, ..., n\}$; hence $\varphi_k(\succsim) \neq \omega_j$ implies $\omega_k \notin C(\succsim_j)$.

$2-$*Efficiency.* It follows by the definition of priority rule.

Unfortunately, the positive results obtained in pairwise exchanges in the age-based environment vanish if we admit larger exchanges.

Theorem 3. *Let* $3 \leq k \leq n-1$. *No rule* $\varphi : \mathcal{E} \to \mathcal{A}^k$ *satisfies* individual rationality*,* k-efficiency*, and* strategy-proofness*.*

Proof. It is just a matter of checking. The preference profiles used in the second step of the proof of Theorem 1 belong to $\mathcal{E}$. Simply consider that $\{\omega_1, \omega_k\} = Y$.

On the other hand, it is easy to extend the definition of age-based priority rules in such a way that we might have more than two classes of indifferences over the set of compatible kidneys of each patient . Moreover, when each indifference class consists of a single donor (hence, the partition induces a strict preference of the set of donors) it turns out that the age-based priority rule (there is only one age-based priority) is the unique pairwise efficient and individually rational rule that satisfies *strategy-proofness*. (Denote such preferences by $\mathcal{E}'$.)

Theorem 4. *Let $\varphi : \mathcal{E}' \to \mathcal{A}^2$. The rule φ satisfies* strategy-proofness,individual rationality*, and* 2-efficiency *if and only if φ is the age-based priority rule.*

Proof. By construction, applying the age-based priority rule in $\mathcal{E}'$ we can find always an assignment in the core. Moreover, it is easy to check that there cannot be any other assignment in the core. It is easy to see that $\mathcal{E}'$ satisfies the conditions on the domain of preferences of **?** (Sönmez: "essentially single-valued cores"). Hence, there exists a rule φ that satisfies *strategy-proofness*, *individual rationality*, and efficiency. Because preferences are strict, $\mathcal{E}'$ satisfies the conditions on the domain of preferences of Takamiya, and φ always selects the unique core assignment.

6 Multiple Donors

Up to now, we only analyze the case in which each patient has a unique donor, and therefore the patient simply reports what is her set of compatible kidneys. Now, we look at the case in which patients may have several potential donors, and this is private information. Of course, such information is valuable for the Transplant Coordinating Center, and needs to be elicited from the patients. In order to analyze patients incentives to manipulate via the introduction or (withdrawn) of potential donors, we need to incorporate the set of donors as an argument of the kidney assignment rule. Let Ω_i denote the set of potential donors of patient i. Let $\Omega \equiv \times_{i \in N} \Omega_i \subset W$ denote the set of kidneyes available for transplantation and W the set of all possible sets of potential donors We assume that preferences in $\mathcal{E}$ are defined over all potential donors. Hence, we define a generalized (kidney assignment) rule as a mapping $\Phi : \mathcal{E} \times W \to A^2$. *When the set of patient $i's$ donors is not a singleton we write $\omega_{ik} \in \Omega_i$* to indicate a specific donor k among patient i's donors*, but when it does not generate confusion we simply write ω_i* to denote a patient $i's$ donor. An ***assignment*** a is an n-tuple of pairs $a = [(1, \omega), \ldots, (n, \omega')]$ such that $i)$

for each $i, j \in N$, $i \neq j$ and each $\omega, \omega' \in \Omega$, if $(i, \omega), (j, \omega') \in a$, then $\omega \neq \omega'$; $ii)$ for each $i \in N$, if for some $j \in N$, $a_j \in \Omega_i$, then $a_k \notin \Omega_i$ for all $k \neq j$. This second requirement imposes that for each patient at most one donor donates her/his kidney. Let $\mathcal{A}$ be the set of all assignments.

The notion of strategy-proofness has to be properly modified since now we require not only that patients have not the incentive to manipulate the information over their preferences but also that they have not incentive to manipulate the set of their potential donors.

Group Strategy–proofness. For each $T \subseteq N$, $\succsim \in \mathcal{D}$, $\Omega \subset \mathcal{W}$, $\succsim'_T \in D_T$ and $\Omega'_T = \times_{i \in T} \Omega'_i$,

$$\text{for each } i \in T \quad \varphi_i(\succsim, \Omega) \succsim_i \varphi_i(\succsim'_T \Omega'_T; \succsim_{N \setminus T}, \Omega_{N \setminus T}).$$

Donor Strategy–proofness. For each $T \subseteq N$, $\succsim \in \mathcal{D}$, $\Omega \subset \mathcal{W}$, and $\Omega'_T = \times_{i \in T} \Omega'_i$,

$$\text{for each } i \in T \quad \varphi_i(\succsim, \Omega) \succsim_i \varphi_i(\succsim_T \Omega'_T; \succsim_{N \setminus T}, \Omega_{N \setminus T}).$$

There are alternative ways to extend age-based priority rules to the multiple donor case. Here, we argue that in order to avoid manipulability, it is necessary that the rule should be focused on assigning the young donors first rather than given priorities to patients with young donors.

Example 2. *Let $N = \{1, 2, 3\}$, $\omega_1 = Y$. Assume that the generalized rule Φ assigns priority to patients with at least one young donor. Consider the case in which each patient has only one donor. Let $\succsim \in \mathcal{E}$ be such that $C(\succsim_1) = \{\omega_2, \omega_3\}$ and $C(\succsim_2) = C(\succsim_3) = \{\omega_1\}$. Assume that the priority ranking is the natural ordering. Then, $\Phi_1(\succsim, \Omega) = \{\omega_2\}$, and $\Phi_3(\succsim, \Omega) = \{\omega_3\}$. In this case, patient 3 manipulates by presenting a new potential donor $\omega'_3 \in Y$ such that $\omega'_3 \notin C(\succsim_1)$, and $\omega'_3 \notin C(\succsim_2)$. The introduction of this "dummy" donor allows patient 3 to be lifted in the priority ranking, in such a way that $\Phi_3(\succsim, \Omega \cup \{\omega'_3\}) = \omega_1$. With multiple donors, it is necessary to define a multi-stage mechanism in order to maintain the non-manipulability of the mechanism. The previous lemma implies that* *patients with young donors have priority over patients with mature donors. However, a patient could obtain a better outcome by introducing "dummy" young donors, just for the sake of improving her priority ranking but exchanging her initial mature donor. So, in the first stage, the priority mechanism tries to match pairs of patients with young donors. Once any matching only involving young donors is performed, in the second stage unmatched patients are assigned using a priority ranking that favor patients whose young donor is compatible with another patient.*

A generalized age-based priority rule works as follows:

Consider an assignment problem, $(\succsim, \Omega)$. Let $N_1 = \{i \in N \mid \exists \omega_i \in Y\}$

Let us proceed in three stages, and consider the priority ordering $(1, 2, \ldots, n_1)$ with $n_1 = |N_1|$ over the patients in N_1.

STAGE 1

- Let $\mathcal{M}^0 = \mathcal{I}(\succsim |_{N_1})$ where $\mathcal{I}(\succsim |_{N_1})$ denote the set of all individually rational pairwise assignments when the population is N_1 such that for all $i \in N_1$ either $a_i \in Y \cap C(\succsim_i)$ or $a_i = \omega_i$.

- For each $k \leq n$ with $k \in N_1$ let $\mathcal{M}^k \subseteq \mathcal{M}^{k-1}$ be such that:

$$\mathcal{M}^k = \begin{cases} \left\{a \in \mathcal{M}^{k-1} : a_i \in C(\succsim_i) \cap Y\right\} & if\ \exists a \in \mathcal{M}^{k-1}\ s.t\ a_i \in C(\succsim_i) \cap Y \\ \mathcal{M}^{k-1} & otherwise. \end{cases}$$

  Let $\bar{N}_1 = \{i \in N_1 | \exists a \in \mathcal{M}^{n_1} : a_i \in Y \cap C(\succeq_i)\}$. Note that $\bar{N}_1$ are those patients who have been involved in a pairwise exchange in the first stage, that is pair of patients with mutually compatible young donors. We remove all patients belonging to $\bar{N}_1$. Let $N_2 = \left\{i \in N \backslash \bar{N}_1 \mid \exists \omega_i \in Y\right\}$ and consider the priority ordering over the set of patients in N_2, $(1, 2, \ldots, n_2)$ where $n_2 = |N_2|$

STAGE 2

- Let $\hat{\mathcal{M}}^0 = \mathcal{I}(\succsim |_{N \backslash \bar{N}_1})$ where $\mathcal{I}(\succsim |_{N \backslash \bar{N}_1})$ denote the set of all individually rational pairwise assignments when the population is $N \backslash \bar{N}_1$.

- For each $k \leq n_2$ with $k \in N_2$ let $\hat{\mathcal{M}}^k \subseteq \hat{\mathcal{M}}^{k-1}$ be such that:

$$\hat{\mathcal{M}}^k = \begin{cases} \{a \in \hat{\mathcal{M}}^{k-1} : a_k \in M \cap \Omega_j, a_j \in Y \cap \Omega_k \text{ for some } j \in N \backslash \bar{N}_1\} & if\ \exists a \in \hat{\mathcal{M}}^{k-1} : a_k \in M \cap \Omega_j, \\ \hat{\mathcal{M}}^{k-1} & othe \end{cases}$$

STAGE 3

Finally, let consider a priority ordering over all the agents in $N \backslash \bar{N}_1$ $(1, .., \bar{n})$ where $\bar{n} = \left| N \backslash \bar{N}_1 \right|$and let $\bar{\mathcal{M}}^0 = \hat{\mathcal{M}}^{n_2}$.For each $k \leq \bar{n}$ with $k \in N \backslash \bar{N}_1$ let $\hat{\mathcal{M}}^k \subseteq \hat{\mathcal{M}}^{k-1}$ be such that:

$$\bar{\mathcal{M}}^k = \begin{cases} \{a \in \bar{\mathcal{M}}^{k-1} : a_k \in Y\} & if\ \exists a \in \bar{\mathcal{M}}^{k-1}\ s.t\ a_k \in Y, \\ \{a \in \bar{\mathcal{M}}^{k-1} : a_k \in M\} & if \begin{pmatrix} \nexists a \in \bar{\mathcal{M}}^{k-1}\ s.t\ a_k \in Y,\ and \\ \exists a \in \in\int^{k-1}\ s.t\ a_k \in M \end{pmatrix}; \\ \bar{\mathcal{M}}^{k-1} & otherwise. \end{cases}$$

Theorem *Consider an assignment problem with multiple donors* $(\succeq, \Omega)$.*A generalized age-based priority mechanism* φ *defined on* $E \times W$ *is group-strategy-proof and* 2*-efficient.*

Proof. $2 - efficiency$ directly follows from priority. We prove here that patients have no incentive to manipulate the set of potential donors, because the proof that they have no incentive to manipulate their preferences uses the same arguements as in Theorem 2. Patients in N_1 have not incentive to manipulate since thet get their preferred kidney. Patients who exchange a young donor for a mature one cannot gain since they have priority over patients with mature donors and cannot be part of N_1 because they need to have young donor compatible with some other patients in N_1....

7 Concluding Remarks

In this paper, we have proposed a model that retains the flavor of the formulation by **?** that has been adopted in the design of a kidney exchange clearing–house in New England, but it departs in an important aspect from their model. We assume that patients may have heterogeneous preferences over the set of compatible kidneys, enlarging the relevant domain of preferences, but at the same time, we assume that the social planner may avail with detailed information about patients' preferences. Our first result shows the difficulties to fulfill different forms of incentive compatibility if there are restrictions on the cardinality of feasible exchanges, even if the heteroegenity over the set of compatible kidney is such that each patients may partion this set in at most two classes. Namely some compatible under the assumptions that the setNamely we assume that the age of the donor is an important characteristic in determining the quality of the transplantion. Therefore we assume that patients prefer to receive a kidney from a young donor than from a mature one.

of available information is large. However, positive results are restored if patients are strongly averse to the risk of refusing a transplant of a compatible kidney. Namely, those rules which are strategy-proof and efficient in the dichotomous domain, still satisfy the desired properties when agents have heterogeneous preferences and adopt protective behavior and only pairwise exchanges are feasible. Interestingly, the difficulties return if larger exchanges are admitted. These results have strong policy implications. If our assumptions fit real life situations, then the efficiency gains of making possible cycles larger than pairwise exchanges can be overcome by the impossibility of eliciting truthful information from patients and the inefficiency that could derive from the strategic manipulation of revealed preferences. Since the cost of slackening the feasibility constraints are high (3-feasibility means having six operating rooms and six surgical teams available at the same time), then our work puts some doubts on the economic advantage of these investments for the healthcare service.

In order to conclude, we devote a few lines to sketch some open venues of further research. First of all the assumption on patients' (protective) behavior deserves to be tested by means of controlled questionnaires on the population of patients in the waiting lists. Second, incentive problems in kidney exchange environments have been studied on static model as ours. However, it is evident that kidney transplantation has a dynamic component that has been neglected (**?** is a remarkable exception). It seems a promising line of new research the analysis of dynamic and strategic models where patients and kidneys are available sequentially and simultaneously living donation and kidney exchange are feasible procedures. In the light of the technical difficulties that appear in standard queue–management models, the analysis of protective behavior in such settings is a promising line of investigation.