Submission Number: ASSET2009-09-00267

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Submitted: July 02, 2009.

Paired Kidney Donation and Listed Exchange

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April 2009

Abstract

Deceased-donor and live-donor kidneys are the two sources for transplantation, and these sources are utilized via two different programs. One of these programs, a *paired kidney donation (PKD)*, involves two donor-patient pairs, for each of whom transplantation from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other pair. These two pairs can then exchange donated kidneys. Another possibility is a *paired listed exchange (PLE)*: given two donor-patient pairs, the first donor provides a kidney to a candidate on the deceased-donor waiting list, the first patient receives the kidney of the second donor, and the second patient receives a priority on the waiting list. By allowing only PKD's and PLE's, we characterize the set of kidney exchanges, which include the maximum number of kidney transplants from the set of incompatible patient-donor pairs. This characterization generalizes the Gallai-Edmonds Decomposition Theorem, a well-known result from combinatorial optimization.

Keywords : Kidney exchange, Gallai-Edmonds Decomposition

Journal of Economic Literature **Classification Numbers:** C71, C78, D71, DF8

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†This research has been supported by TUBITAK Grant No. 1001-107K241.

1 Introduction

Transplantation is the preferred treatment for the most serious forms of kidney disease. Unfortunately, there is a considerable shortage of deceased-donor kidneys: as of June 13, 2008, there are 76,313 patients waiting for kidney transplants in the U.S., with the median waiting time of over 3 years, and in 2007, there were only 10,587 transplants of deceased-donor kidneys. The cadaveric kidneys are not the only sources for transplantation. Since healthy people have two kidneys and can remain healthy on one, it is also possible for a kidney patient to receive a live-donor transplant. In 2007, there were 6,038 transplants of live-donor kidneys. Our goal is to characterize the exchanges utilizing these two sources of kidneys as much as possible subject to the constraint that an exchange includes two transplantations.

The two sources of kidneys enable the medical authorities to develop different programs to increase the number of transplantations. One of these programs is a *paired kidney donation (PKD)*. A PKD involves two patient-donor pairs, for each of whom a transplant from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other couple (Rapaport [8], Ross et al. [9, 10]). These two pairs can then exchange donated kidneys. A PKD is depicted in Figure 1.

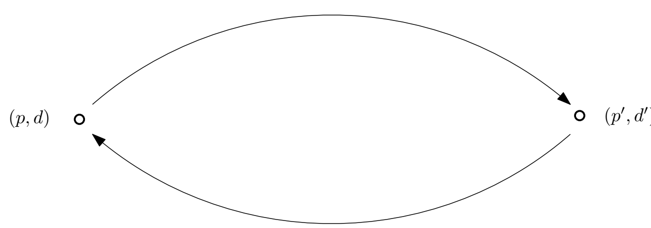


Figure 1: A pairwise kidney donation

Another possibility is a *paired listed exchange (PLE)*. In a PLE, there are two donor-patient pairs: the first donor provides a kidney to a candidate on the deceased-donor waiting list, the first patient receives the kidney of the second donor, and the second patient receives a priority on the waiting list. This improves the welfare of the first patient, but also of the second patient, compared to having a long wait for a compatible cadaver kidney, and it benefits the recipient of the live kidney on the waiting list who benefit from the increase in the kidney supply due to an additional living donor. Through April 2006, 24 listed exchanges have been performed. A PLE is depicted in Figure 2.

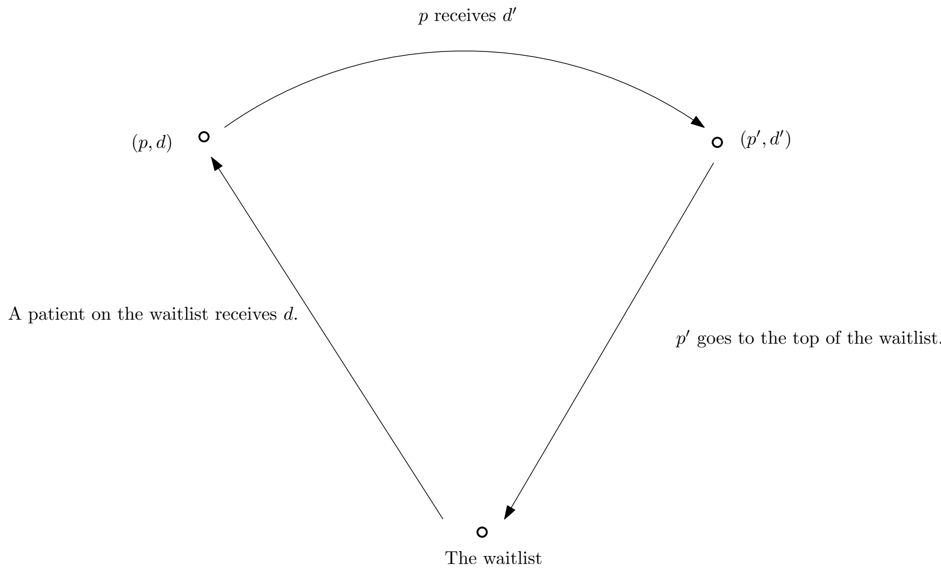


Figure 2: A pairwise listed exchange

Any type of exchange includes multiple transplantations, and these transplantations are carried out simultaneously to avoid conflicts which may arise when a donor gives up after her patient receives a kidney transplant from another donor. But this practice excludes the exchanges with three or more incompatible pairs. Thus, the only available procedures are PKD's and PLE's, and any exchange mechanism constrained in this way, matches pairs. Our goal is to characterize the set of matchings with the maximum number of patients receiving a transplant.

Kidney exchange is matching problem, where the pairs are matched according to the medical compatibilities between them. There are two different interpretations when two pairs are matched: it is either a paired kidney donation, where the pairs exchange the live donors, or a paired listed exchange, where the first patient receives a live donor kidney and the second pair receives a priority on the waiting list. Since the second pair actually could have engaged in a listed exchange by itself, it is the first pair who benefited from the paired listed exchange; the second patient would have received a priority on the waiting list anyway. Thus, while two patients benefit from a paired kidney donation, only one patient benefits from a paired listed exchange, and kidney exchange is a weighted matching problem. We provide a characterization of the matchings with the maximum number of patients receiving a transplant; these are the maximum weight matchings. This result generalizes the result on the maximum cardinality matchings.[1]

[1]The maximum cardinality matching problem is well analyzed in the graph theory literature. More specifically, the Gallai [4, 5]-Edmonds [2, 3] Decomposition Lemma characterizes the set of maximum cardinality matchings. We make use of this result in constructing an efficient exchange.

2 Related Literature

While the transplantation community approved the use of the paired donations and listed exchanges to increase kidney donations, it has provided little guidance about how to organize such exchanges. Roth, Sönmez, and Ünver [11, 12, 13, 14] suggested that, by modeling kidney exchange as a mechanism design problem, integrating the paired donations and listed exchanges may benefit additional candidates. This approach turns out to be very successful and is supported by the medical community. Since then, a centralized mechanism for kidney exchange based on these two protocols has been used in the regional exchange program in New England (The United Network for Organ Sharing-UNOS-Region 1).

Roth, Sönmez, and Ünver [12] assumed dichotomous preferences and considered the constrained kidney exchange problem, in which only the paired kidney donations are allowed. They show that, in the constrained problem, efficient and strategy-proof mechanisms exist. These mechanisms include a deterministic mechanism based on the priority setting that organ banks currently use for the allocation of cadaver kidneys, and a stochastic mechanism motivated by the fairness considerations. The results of Roth, Sönmez, and Ünver [18] on the egalitarian mechanism generalize the corresponding results of Bogomolnaia and Moulin [5] to general (not necessarily bipartite) graphs.[2] This work is tied to the current one as follows: they construct a random mechanism on the set of maximum cardinality matchings in the corresponding graph; the characterization of the maximum cardinality matchings given by the Gallai-Edmonds Decomposition Lemma [2, 3, 4, 5, 7]. While maximum cardinality matchings are defined with respect to regular graphs with uniform edges, our contribution is to generalize this decomposition result to the graphs with weights 1 and 2.

Yılmaz [18] explores how to organize kidney exchange by integrating the multiple ways kidney donations and listed exchanges, under the assumptions of dichotomous preferences of the patients, and that the success rates of transplants from live donors are higher than those from cadavers. He characterizes the set of random matchings, which are Pareto efficient and fair.

Recently, Sönmez, and Ünver [16] extended the Decomposition Theorem by including the compatible pairs as well. They consider the paired kidney donations between the incompatible pairs or between an incompatible pair and a compatible pair. In terms of modeling kidney exchange, and their characterization result, this work is the closest to the current one.

[2]Roth, Sönmez, and Ünver [13] also explore that, for a specific preference profile of the patients (this profile is constructed according to the medical facts on the blood-type compatibilities), when multiple-way paired donations are feasible, three-way kidney donations as well as paired kidney donations will have a substantial effect (and larger than three-way kidney donations have less impact) on the number of transplants that can be arranged.

3 The model

A **pair** consists of a patient and a donor such that the donor cannot medically donate her kidney to the patient of the pair. Let N be the set of all pairs. Given two pairs $n, n' \in N$, n **is compatible with** n' if the donor of n' can medically donate her kidney to the patient of n. For each pair, the donor has the same preferences with the patient; let $\succcurlyeq_n$ denote the preferences of the pair n over the set N. Let $\succ_n$ denote the strict preference relation and $\sim_n$ denote the indifference relation associated with $\succcurlyeq_n$. Let $\succcurlyeq = (\succcurlyeq_n)_{n \in N}$.

Each pair n has dichotomous preferences on N: it is indifferent between all compatible pairs, it indifferent between all incompatible pairs and it strictly prefers a compatible pair to remaining unmatched and remaining unmatched to an incompatible pair. A pair can be matched to a compatible pair, remain unmatched or receive a cadaveric kidney transplantation. This last alternative is possible via a **listed exchange**: In a listed exchange, the patient receives the top priority in the waiting list, and in exchange, his donor donates her kidney to a patient on the waiting list. In a **paired listed exchange**, there are two pairs n, n', such that n is compatible with n', the patient of n receives the kidney of n', the donor of n donates to a patient in the waiting list, and the patient of n' receives the top priority in the waiting list. Due to a medical fact, that a live donor kidney has a substantially higher patient survival and graft survival rates than the cadaveric donor kidney, each pair strictly prefers a compatible pair to a cadaveric kidney.

There are two types of pairs: a **p-pair** prefers being unmatched to a cadaveric kidney transplantation and an **l-pair** prefers a cadaveric kidney transplantation to being unmatched. The sets of p-pairs and the l-pairs are denoted by N_p and N_l, respectively and they partition the set N.

A **feasible exchange matrix** $R = [r_{x,y}]_{x,y \in N}$ identifies all feasible paired kidney donations and paired listed exchanges where

$$r_{x,y} = \begin{cases} 2 & \text{if } y \in N \setminus \{x\}, \text{ and } x, y \text{ are mutually compatible} \\ 1 & \text{if } x \text{ is compatible with } y,\ y \text{ is not compatible with } x, \text{ and } y \in N_l \\ 0 & \text{otherwise.} \end{cases}$$

For each $x, y \in N$ with $r_{x,y} = 2$, we refer the pair (x, y) as a **feasible paired kidney donation**. For $x, y \in N$ with $r_{x,y} = 1$, we refer the pair (x, y) as a **feasible paired listed exchange** where the patient of x receives the kidney from the donor of y.

A **kidney exchange problem** (or simply a **problem**) (N, R) consists of a set of pairs and its feasible exchange matrix. Given a problem (N, R) and $N' \subseteq N$, the reduced problem is denoted by $(N', R|_{N'})$, where $R|_{N'}$ is the reduced matrix of R on N'.

A **matching** μ is a set of mutually feasible *paired kidney donations* and *paired listed* *exchanges*. For each matching μ, the set of paired kidney donations in μ is denoted by μ_2 and the set of paired listed exchanges in μ is denoted by μ_1; thus, $\mu = \mu_1 \cup \mu_2$. For each matching μ, $(x, y) \in \mu_2$ means that the patient of each pair receives a kidney from the donor of the other pair; $(x, y) \in \mu_1$ means that the patient of x receives a kidney from the donor of y, the donor of x donates to a patient on the waiting list, the patient of y receives the top priority in the waiting list, and $y \in N_l$. For a problem (N, R), let $\mathcal{M}(N, R)$ denote the set of all matchings.

Observe that an l-pair can always receive the top priority in the waiting list by simply accepting to be in a listed exchange. Thus, comparing two different matchings, only the patients, who receive a transplant from a live donor matter. For each matching μ, let T^μ denote the set of all pairs who receive a transplant from a live donor. Formally,

$$T^\mu = \{x \in N : \mu_2(x) \neq x \text{ or } (x, y) \in \mu_1 \text{ where } x \neq y\}$$

4 Maximal matchings

In our model, the interpretation of the sets N_p and N_l is the following: There is a set of pairs N_p, each of whom expects to receive a transplant from a live donor, and they form a pool for paired kidney donation. On the other hand, there is a set of pairs N_l, each of whom accepted to be in a listed exchange to receive the top priority in the waiting list. These two sets are integrated so that both groups of pairs will benefit from the extended set of feasible feasible paired kidney donations and feasible paired listed exchanges. For example, let $N_p = \{x\}$ and $N_l = \{y\}$ such that $r_{x,y} = 1$. If these two pairs are considered separately, then the pair x remains unmatched, and the pair y receives the top priority in the waiting list. On the other hand, if, as our model suggests, these two groups are considered together, then the pair x receives a transplant from a the donor of y and as before, the pair y receives the top priority in the waiting list.

For each $\mu, \mu' \in \mathcal{M}$, μ **Pareto-dominates** μ' if, for each $x \in N$, $\mu(x) \succcurlyeq_x \mu'(x)$, and for some $x \in N$, $\mu(x) \succ_x \mu'(x)$. A matching $\mu \in \mathcal{M}$ is **Pareto efficient** if no other matching Pareto dominates μ. For a problem (N, R), let $\mathcal{E}(N, R)$ denote the set of Pareto efficient matchings.

When there are no l-pairs, a well-known result, the Gallai-Edmonds Decomposition Theorem, characterizes the structure of Pareto efficient matchings, and the same number of pairs are matched at each Pareto efficient matching. (Observe that when there no l-pairs, only the paired kidney donations are feasible, and there is no paired listed exchange in a matching.) For a kidney exchange problem with p-pairs and l-pairs, what is critical is the set of pairs who receive a transplant from a live donor, not the set of pairs who are matched, and for this general case, this result does not extend and the number of pairs, who receive a transplant from a live donor, may be different in different Pareto efficient matchings.

Example 1: Let $N_p = \{x\}$ and $N_l = \{u, v\}$. The feasible exchange matrix R is such that $r_{x,u} = 1$ and $r_{u,v} = 2$. Observe that there are two efficient matchings, $\mu = \{(x, u)\}$ and $\mu' = \{u, v\}$, where $|T^{\mu}| = 1$, and $\left|T^{\mu'}\right| = 2$.

Observe that in Example 1, the number of p-pairs as well, who receive a transplant from a live donor, is different in different Pareto efficient matchings.

What Example 1 makes clear is that some Pareto efficient matchings can be improved in terms of the number of pairs who receive a transplant. A matching μ has the **maximum number of transplants** if there is no other matching μ' such that $\left|T^{\mu'}\right| > |T^{\mu}|$. Also, the transplantation centers' preferred exchange is the paired kidney donation. Thus, it is plausible to minimize the number paired listed exchanges, while maximizing the number of transplant. A matching is called **maximal** if it has the maximum number of transplants and it has the maximum number of paired kidney donations in the set of the matchings with the maximum number of transplants. From now on, we focus on this particular property. Given a problem (N, R), let $\mathcal{E}^m(N, R)$ denote the set of maximal matchings.

The exchange mechanism used in practice considers the pool of the p-pairs separately and matches the pairs in a Pareto efficient way. Note that, as mentioned above, each Pareto efficient matching maximizes the number of transplants (each such is maximal as well); maximality and Pareto efficiency coincide in this specific class of problems. In our model, the l-pairs are integrated to the pool of the p-pairs so that this enhances the number of transplants. The foremost important question is whether the l-pairs may have a negative externality on the welfare of the p-pairs, if the transplantation center insists on the maximality property. More specifically, suppose a Pareto efficient matching is fixed for the group of p-pairs only. Let T be the set of pairs who receive a transplant in this matching. Then, after the integration of the l-pairs to the pool of p-pairs, does there exist a maximal matching in this new problem, so that all the patients in T receive a transplant in this maximal matching? As our first result shows, the answer is yes.

Proposition 1 *Let $(N_p \cup N_l, R)$ be a problem. Let $\mu \in \mathcal{E}^m(N_p, R|_{N_p})$. Then, there exists a matching $\mu' \in \mathcal{E}^m(N_p \cup N_l, R)$ such that $T^{\mu'} \supseteq T^{\mu}$.*

5 The structure of maximal matchings

Our goal here is to characterize the set of maximal matchings. Our model is built on integrating the groups of p-pairs and l-pairs. Thus, we focus on the static problem, in which the sets N_p and N_l are given; there are no strategic issues such as a patient revealing truthfully or not whether he is an $l-$patient.

For each problem (N, R), define

$$D(N, R) = \{x \in N : \exists \mu \in \mathcal{E}^m(N, R) \text{ s.t. } \mu(x) = x\}.$$

Let $D_1(N, R)$ and $D_2(N, R)$ be the sets of pairs who are in $D(N, R)$ and part of only paired listed exchanges and of only paired kidney donations, respectively, in any maximal matching. Let $D_{1,2}(N, R) = D(N, R) \setminus (D_1(N, R) \cup D_2(N, R))$.

Let $A_1(N, R)$ be the set of pairs who are part of only paired listed exchanges in each maximal matching and have a compatibility with at least one pair in $D(N, R)$. Let $C_1(N, R)$ be the set of pairs who are part of only paired listed exchanges in each maximal matching, which are not in $A_1(N, R)$.

Let $A_{1,2}(N, R)$ be the set of pairs who are part of a paired listed exchange in some maximal matchings and part of a paired kidney donation in the remaining maximal matchings, who have a compatibility with at least one pair in $D_1(N, R) \cup D_{1,2}(N, R)$.

Let $C_{1,2}(N, R)$ be the set of pairs who are part of a paired listed exchange in some maximal matchings and part of a paired kidney donation in the remaining maximal matchings, who are not in $A_{1,2}(N, R)$.

Theorem 1 *Let (N, R) be a problem. Then, in any maximal matching,*

1. *each pair in $A_1(N, R)$ is matched to a pair in $D(N, R)$,*
2. *each pair in $C_1(N, R)$ is matched to a pair in $C_1(N, R) \cup C_{1,2}(N, R)$;*
3. *if the subgraph induced by $G - A_1(N, R)$ contains a component N' in $D(N, R)$, then*
   (a) this component is such that, for each $x \in N'$, $N' - x$ has a perfect matching,
   (b) any maximal matching contains a near-perfect matching of this component,
   (c) any maximal matching matches at most one pair of this component to a pair in $A_1(N, R)$.

6 Appendix

6.1 Preliminaries on graphs

A problem (N, R) can be represented by a weighted graph $G = (V, E, w)$, where V is the set of vertices, E is the set of edges, and w is a weight function, $w : E \rightarrow \{1, 2\}$. The graph representation of a problem (N, R) is obtained as follows: Each pair u is a vertex, thus $V = N$.

Let $u, v \in V$ be two vertices. If $r_{u,v} = 2$, then the set E contains the edge uv, the weight of which is 2. If $r_{u,v} = 1$, then the set E contains the directed edge from u to v, the weight of which is 1. An edge with weight 1 is called a **thin edge**; an edge with weight 2 is called a **thick edge**. For a weighted graph $G = (V, E, w)$, and $E' \subseteq E$, let $w(E') = \sum_{uv \in E'} w(uv)$.

Let $G = (V, E, w)$ be a weighted graph. For a subgraph $H \subseteq G$, let H_2 be the induced graph of H on the thick edges. The set of thin and thick edges are denoted by $E_1(G)$ and $E_2(G)$, respectively. Note that a matching μ is a subset of the edges such that no two edges meet at a common vertex. Let $\mathcal{M}(G)$ denote the set of all matchings. Let $\nu(G) = \underset{\mu \in \mathcal{M}(G)}{Max}\, w(\mu)$. If the weights are uniform, then $\nu(G)$ is called the **matching number** of G.

Definition 1 *A vertex is **free** with respect to a matching μ if it is not incident with any edge in μ.*

Definition 2 *A path (or a cycle) is **alternating** with respect to a matching μ if its edges are alternately in μ and not in μ.*

Definition 3 *An **augmenting path** is an alternating path between free vertices. An augmenting path with respect to a matching μ is called an $\mu-$**augmenting path**.*

If the weights of the edges are uniform, then a well-known result in matching theory characterizes the condition for the maximality of a matching.

Theorem 2 *Given all the edges have the same weight, a matching μ is a maximal matching if and only if there does not exist an $\mu-$augmenting path.*

6.2 Proofs

First, we extend Theorem 2 to the graphs consisting of both thick and thin edges. Let $\oplus$ denote the set difference operator, for $E', E'' \subseteq E$, $E' \oplus E'' = (E' \setminus E'') \cup (E'' \setminus E')$.

Lemma 1 *A matching μ is maximal if and only if $|\mu_2| = \nu(G_2)$ and there does not exist an $\mu-$augmenting path P with $E_2(P \cap \mu) = E_2(P \setminus \mu)$.*

Proof. (Only if) Clearly, $|\mu_2| \leq \nu(G_2)$. Suppose the inequality is strict. Then, by Theorem 2, there is an μ_2-augmenting path in G_2, say P. If both of the end vertices in P are free in μ, then $w(\mu \oplus P) = w(\mu) + 2$, which contradicts with μ having the maximum weight. Suppose only one of the end vertices, say u, is covered in μ, say $ux \in \mu$. Since P is an μ_2-augmenting path in G_2, the edge ux is thin. Then, $w(\mu \oplus (P + ux)) = w(\mu) + 1$, which again contradicts with μ having the maximum weight. Suppose both end vertices in P, say u and v, are covered

by the (thin) edges in G, say by ux and vy. Then, $w(\mu \oplus (P + ux + vy)) = w(\mu)$ and the matching $\mu \oplus (P + ux + vy)$ has one more thick edge than the matching μ, which contradicts with maximality of μ. Thus, $|\mu_2| = \nu(G_2)$. Now, suppose there exists an $\mu-$augmenting path P such that $E_2(P \cap \mu) = E_2(P \setminus \mu)$. Since the end vertices of the path P are free, the set $P \oplus \mu$ is another matching. Since the matching $P \oplus \mu$ contains one more edge than μ and both μ and $P \oplus \mu$ have equal number of thick edges, $w(P \oplus \mu) = w(\mu) + 1$, which contradicts with the matching μ having the maximum weight.
(If) Suppose μ is a non-maximum weight matching with $|\mu_2| = \nu(G_2)$. Let μ' be a maximal matching. Since, in a matching, no two edges meet at a common vertex, each vertex is incident with at most one vertex in μ and one vertex in μ'. Thus, the set $\mu \oplus \mu'$ contains connected components of the form of either an $\mu-$alternating path or an $\mu-$alternating cycle. Since $w(\mu') > w(\mu)$, one of these alternating paths or cycles, say H, is such that $w(H \cap \mu') > w(H \cap \mu)$. Suppose there exists such an alternating path, say P. Since μ contains the maximum possible number of thick edges, $E_2(P \cap \mu) \geq E_2(P \cap \mu')$. Otherwise, $\mu \oplus P$ contains more thick edges than μ, which is a contradiction. Now, suppose $|P \cap \mu'| = |P \cap \mu|$ or $|P \cap \mu'| = |P \cap \mu| - 1$. Since $E_2(P \cap \mu) \geq E_2(P \cap \mu')$, in either case, $w(P \cap \mu') \leq w(P \cap \mu)$. Thus, P is such that $|P \cap \mu'| = |P \cap \mu| + 1$ and $E_2(P \cap \mu) = E_2(P \cap \mu')$. Note that this implies that P is an $\mu-$augmenting path with $E_2(P \cap \mu) = E_2(P \setminus \mu)$. Now, suppose there exists an alternating cycle, say C, such that $w(C \cap \mu') > w(C \cap \mu)$. Since each vertex is incident to at most one vertex in μ and one vertex in μ', the cycle C is an even size cycle. But this is impossible since $E_2(C \cap \mu) \geq E_2(C \cap \mu')$ implies $w(C \cap \mu) \geq w(C \cap \mu')$. ■

PROOF of PROPOSITION 1:
Let $(N_p \cup N_l, R)$ be a problem. Let $\mu \in \mathcal{E}^m(N_p, R|_{N_p})$. Let $\mu' \in \mathcal{E}^m(N_p \cup N_l, R)$ such that $T^{\mu'} \not\supseteq T^{\mu}$. Then, there exists $x \in T^{\mu} \setminus T^{\mu'}$. Since x is covered by μ but not by μ', the set $\mu \oplus \mu'$ contains a path P starting at x. Let $P = x, x_1, x_2, \ldots, x_k, y$ be this path. Observe that since μ is a matching for the problem $(N_p, R|_{N_p})$ and all the vertices $x_1, x_2, \ldots, x_k$ are covered by μ and μ', these vertices are in N_p. Moreover, according to the definition of R, $r_{a,b} = 1$ implies that $b \in N_l$. Thus, all the edges $x_1x_2, x_2x_3, \ldots, x_{k-1}x_k$ in P are thick. If y is covered by μ, then P is an $\mu'-$augmenting path: the matching $P \oplus \mu'$ contains more thick edges and has a higher weight than μ', contradicting the maximality of μ'. Thus, y is not covered by μ but covered by μ'. Then, the matching $P \oplus \mu'$ has a weight at least as $w(\mu')$, has the number of thick edges at least as $|\mu'_2|$. Thus, by the maximality of μ' the edge x_ky is thick, and the matching $P \oplus \mu'$ is maximal as well. Observe that the matching $P \oplus \mu'$ covers all the vertices $x, x_1, x_2, \ldots, x_k$, but not y. Since y is not covered by μ, we obtained a maximal matching $P \oplus \mu'$ such that $T^{P \oplus \mu'} = (T^{\mu'} \cup \{x\}) \setminus \{y\}$. By applying the same argument recursively, a maximal matching is obtained, where it covers all the vertices in T^{μ}.

PROOF of THEOREM 1:

Lemma 2 *If $u \in D_1(G)$, then $\delta(u) \subseteq E_1(G)$.*

Proof. Let $u \in D_1(G)$. Suppose there exists a thick edge incident with u in G. Since $u \in V(G_2)$, either $u \in D(G_2)$ or $u \in V(G_2) \setminus D(G_2)$. If the latter, then by Lemma 1, $u \in C_2(G)$, which is a contradiction. Thus, $u \in D(G_2)$. Let M be a maximal matching missing u. By the GED Theorem and Lemma 1, each neighbor of u in $V(G_2)$ is covered by a thick edge in M. Let v be such an edge and let $vw \in M$. Then, clearly $(M \setminus \{vw\}) \cup \{uv\}$ is another maximal matching. But this contradicts with $u \in D_1(G)$. ■

Lemma 3 *Let G be any graph. Let $u \in A_1(G)$. Then, (i) $D_1(G-u) = D_1(G)$, (ii) $D_{1,2}(G-u) \subseteq D_{1,2}(G)$, (iii) $D(G-u) = D(G)$, (iv) $A_1(G) - \{u\} = A_1(G-u)$.*

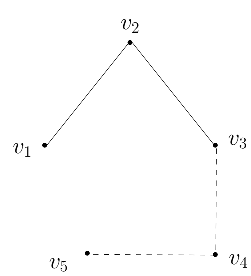
Proof.

(i) First, we show that $D_1(G) \subseteq D_1(G-u)$. Let $v \in D_1(G)$ and let M be a matching missing v. Consider $M - u$, that is the matching obtained by removing the edge incident with u in M. Clearly, $M - u$ misses v. Since the weight of $M - u$ is $w(M) - 1$ and $u \in A_1(G)$, the matching $M - u$ is a maximal matching in $G - u$. By Lemma 2, $v \in D_1(G)$ implies $\delta(u) \subseteq E_1(G)$. Thus, $v \notin D_{1,2}(G-u)$. Thus, $v \in D_1(G-u)$.
Now, we show that $D_1(G-u) \subseteq D_1(G)$. Let $v \in D_1(G-u)$. First note that $v \in D_{1,2}(G) \cup D_2(G)$ is not possible. Suppose on the contrary that there exists a maximal matching in G, which covers v by a thick edge. Since $u \in A_1(G)$, this maximal matching does not contain uv. By removing the edge that covers u, one obtains a maximal matching in $G-u$, which covers v by a thick edge but since $v \in D_1(G-u)$, it is a contradiction. Now, let M be a maximal matching in $G-u$ missing v. Let v' be a vertex in $D_1(G) \cup D_{1,2}(G)$ adjacent to u. Let M' be a maximal matching in G missing v'. If $v' = v$, then $v \in D_1(G)$. Suppose $v' \neq v$. If M' misses v, then $v \in D_1(G)$. So suppose M' covers v. Consider $M \cup M'$. Since both M and M' are matchings, $M \cup M'$ contains paths and cycles. Since M misses v, $M \cup M'$ contains a path P starting at v with an edge in M'. Suppose P ends with an edge of M. Then, since M is in $G-u$, P does not cover u. We claim $w(P \cap M) = w(P \cap M')$. Suppose $w(P \cap M) > w(P \cap M')$. Then, $M' \oplus P$ is another matching in G such that $w(M' \oplus P) > w(M')$, contradicting with the maximality of M'. Similarly, if $w(P \cap M') > w(P \cap M)$, then it contradicts with the maximality of M. Thus, $w(P \cap M) = w(P \cap M')$. Also, by Lemma 1 and since $u \in A_1(G)$, the number of thick edges in M must be equal to the number of thick edges in M'. Thus, $P \cap M$ contains the same number of thick edges as $P \cap M'$. (Since otherwise either $M \oplus P$ contains more thick edges than M, both having the same weight or $M' \oplus P$ contains more thick edges

than M', both having the same weight; this contradicts with the maximality of the number of the thick edges.) But then $M' \oplus P$ is another maximal matching in G. Moreover, it misses v. Thus, $v \in D_1(G)$.

Suppose P ends with an edge of M'. We claim P ends at u. Suppose not. Then, if $w(P \cap M) > w(P \cap M')$, then $M' \oplus P$ is another matching in G such that $w(M' \oplus P) > w(M')$, contradicting with the maximality of M'. Similarly, if $w(P \cap M') > w(P \cap M)$, then it contradicts with the maximality of M. Thus, $w(P \cap M) = w(P \cap M')$. Then, since $P \cap M'$ contains more edges than $P \cap M$, $P \cap M$ contains one more thick edge than $P \cap M'$. But then, $M' \oplus P$ has weight $\nu(G)$ but $|M_2'| < \nu(G_2)$, contradicting with the maximality of M'. Thus, P ends at u. Now, consider $P' = P \cup uv'$. If $w(P' \setminus M') > w(P' \cap M')$, then $w(P' \oplus M') > w(M')$, which contradicts with the maximality of M'. Thus, $w(P' \setminus M') \leq w(P' \cap M')$. Suppose $w(P' \setminus M') < w(P' \cap M')$. Let x be the vertex such that $ux \in M'$. Since $u \in A_1(G)$, $w(ux) = w(uv') = 1$. Define $P'' = P' \setminus \{xu, uv'\}$. Now, since $w(P' \setminus M') < w(P' \cap M')$, we have $w(P'' \setminus M') < w(P'' \cap M')$. But then, $P'' \oplus M$ is a matching in $G - u$, and moreover $w(P'' \oplus M) > w(M)$, contradicting with the maximality of M. Thus, $w(P' \setminus M') = w(P' \cap M')$. But then, $M' \oplus P'$ is a matching in G. Moreover, since P' is an even path and $w(M' \oplus P') = w(M')$, $M' \oplus P'$ has the same number of thick edges as in M'. Thus, $M' \oplus P'$ is a maximal matching in G. Since $M' \oplus P'$ does not cover v, by definition, $v \in D_1(G)$.

(ii) The proof is essentially the same of the proof of part (i). Note that in general, $D_{1,2}(G) \not\subseteq D_{1,2}(G-u)$. This can be easily seen via a simple example. Consider the following graph:



Here, $D_{1,2}(G) = \{v_3\}$, $A_2 = \{v_4\}$, and $D_1 = \{v_5\}$. Also, $D_{1,2}(G - v_4) = \emptyset$. Now, let $v \in D_{1,2}(G) \setminus D_{1,2}(G-u)$. Let M and M' be two maximal matchings in G missing v and covering v by a thick edge, respectively. Since $u \in A_1(G)$, both $M - u$ and $M' - u$ are maximal in $G - u$. By part (i), $v \notin D_1(G-u)$. Thus, $v \in D_2(G-u)$. We also claim that the only vertex incident to v in $A_1(G)$ is u. Otherwise suppose v is incident to $v' \in A_1(G)$ with $v' \neq u$. Consider M again. Note that M does not contain uv'. Also, $M - u$ covers v' by a thin edge, say

xv' and misses v. But then, $M - xv' + vv'$ is also maximal in $G - u$ contradicting with $v \in D_2$.

(iii) Let M be a maximal matching in G missing v. Then, since $M - u$ is maximal in $G - u$ and misses v, $v \in D(G - u)$. Thus, $D(G) \subseteq D(G - u)$. By essentially the same proof as in part (i), $D_2(G - u) \subseteq D(G)$. Then, by the parts (i) and (ii), $D(G - u) \subseteq D(G)$.

(iv) Let $v \in A_1(G) - \{u\}$. Suppose that in all the maximal matchings in G, u is matched to v. Let v' be a vertex in $D_1(G) \cup D_{1,2}(G)$, which is incident to v. Consider a maximal matching M' missing v'. Since $uv \in M'$, the matching $M' - uv + vv'$ is also maximal in G and misses u, which contradicts with $u \in A_1(G)$. Thus, there exists at least one maximal matching in G such that u and v are not matched. Let M be such a maximal matching in G and consider the matching in $M - u$ in $G - u$. Since $u \in A_1(G)$ and $w(M - u) = w(M) - 1$, the matching $M - u$ is maximal in $G - u$. Moreover, it covers v with a thin edge. Thus, $v \in D_1(G - u) \cup D_{1,2}(G - u) \cup A_1(G - u) \cup A_{1,2}(G - u) \cup C_1(G - u) \cup C_{1,2}(G - u)$. By part (i) and (ii) above, $v \notin D_1(G - u) \cup D_{1,2}(G - u)$. Suppose $v \in C_1(G - u) \cup C_{1,2}(G - u)$. By part (i) and (ii), this is possible only if the set of neighbors of v in $D_1(G) \cup D_{1,2}(G)$, say V', is in $D_{1,2}(G) \setminus D_{1,2}(G - u)$. But as we have shown in part (ii), any vertex in V' has only one incident vertex in $A_1(G)$, which is u. Thus, $v \notin C_1(G - u) \cup C_{1,2}(G - u)$. Thus, $A_1(G) - \{u\} \subseteq A_1(G - u) \cup A_{1,2}(G - u)$.[3] Now, suppose there exists $x \in A_{1,2}(G - u)$ such that $x \in A_{1,2}(G - u)$. Let M be a maximal matching in $G - u$ covering x by a thick edge. Let $y \in D_1(G) \cup D_{1,2}(G)$ be a neighbor of u and M' be a maximal matching missing in G missing y. Also, let $uv \in M'$. Note that, $M' - uv$ is maximal in $G - u$. If M misses y, then $M + uy$ is maximal and it covers x by a thick edge, which contradicts with $x \in A_1(G)$. Thus, M covers y. Then, $M \oplus (M' - uv)$ contains a path P starting at y with an edge of M. Suppose the path P covers v. Then, since $M' - uv$ misses v, P should end at v with an edge of M, which, by Lemma 1, contradicts with the maximality of $M' - uv$ in $G - u$. Now, suppose the path does not cover v. But then, $M' \oplus P$ is maximal in G, covering x by a thick edge, which contradicts with $x \in A_1(G)$. Thus, $A_1(G) - \{u\} \subseteq A_1(G - u)$. Now, we show that $A_1(G - u) \subseteq A_1(G) - \{u\}$. Suppose this is not true. Let $x \neq u$ such that $x \in A_1(G - u)$ but $x \notin A_1(G)$. Since for any maximal matching M missing a vertex $x \neq u$, $M - u$ is also missing x and maximal in $G - u$, all maximal matchings in G cover x. Similarly, since for any maximal matching M covering a vertex $x \neq u$ by a thick edge, $M - u$ also covers x by a thick edge and maximal in $G - u$, no maximal matching in G covers x by a thick edge. Thus, $x \in A_1(G) \cup C_1(G)$. Suppose $x \in C_1(G)$. Then, by definition of $C_1(G)$, x is not incident to

[3] This can be shown as follows as well: We claim that in $G - u$, each vertex in V' is missed by at least one maximal matching. Let $v'' \in V'$. Let M'' be a maximal matching in G missing v''. Since $M'' - u$ is maximal in $G - u$, v'' is missed by at least one maximal matching in $G - u$, in particular by M''. Now, since $v \in C_1(G - u)$, M'' covers v by a thin edge, say by ux. But then the matching $M'' - ux + uv''$ is also maximal in $G - u$, contradicting with $v'' \notin D_{1,2}(G - u)$.

any vertex in $D_1(G) \cup D_{1,2}(G)$. But then, since by parts (i) and (ii), $D_1(G-u) = D_1(G)$ and $D_{1,2}(G-u) \subseteq D_{1,2}(G)$, x is not incident to any vertex in $D_1(G-u) \cup D_{1,2}(G-u)$ neither. But this contradicts with $x \in A_1(G-u)$. Thus, $A_1(G-u) \subseteq A_1(G) - \{u\}$. ■

Lemma 4 *Let $u \in A_1(G)$. Then, in each maximal matching, u is matched with a vertex in $D_1(G) \cup D_{1,2}(G)$.*

Proof. Let $u \in A_1(G)$ and M be a maximal matching. Suppose M contains uv where $v \in A_1(G)$. Since $u \in A_1(G)$, the matching $M - uv$ is maximal in $G - u$. By the part (iv) of Lemma 3, $v \in A_1(G-u)$. But, this contradicts with $M - uv$ being maximal and missing v in $G-u$. Now, suppose M contains uv where $v \in C_1(G)$. Since $u \in A_1(G)$, the matching $M-uv$ is maximal in $G-u$. Let $v' \in D_1(G) \cup D_{1,2}(G)$ be a vertex incident with u. The matching M covers v', since otherwise, $M - uv + vv'$ is maximal in G and misses v, contradicting with $v \in C_1(G)$. Since $v' \in D_1(G) \cup D_{1,2}(G)$, there exists a maximal matching missing v'; let M' be such a matching. Consider $(M-uv) \oplus M'$; it contains alternating paths and cycles. Since v' is covered by $M - uv$ and missed by M', $(M-uv) \oplus M'$ contains a path P starting at v'. Suppose P does not cover u and v. Then, by Lemma 1, the maximality of $M - uv$ (in $G-u$) and M' (in G) implies that P is not an augmenting path. Thus, the matching $(M-uv) \oplus P$ is maximal in G and misses v'. But then, the matching $(M \oplus P) - uv + vv'$ is maximal in G and misses v, which contradicts with $v \in C_1(G)$. Thus, the path P covers either u or v. Suppose P covers u. Since $M - uv$ does not cover u, P ends at u with an edge ux of M'. Since $u \in A_1(G)$, $M' - ux$ is also maximal in $G-u$. Now, the alternating path $P - ux$ both starts and ends with an edge of $M - uv$. Thus, $P - ux$ is an augmenting path, which by Lemma 1 contradicts with the maximality of either $M - uv$ or $M - ux$ in $G-u$. Now, suppose P covers v. Note that $M - uv$ misses v. Now, the path P is in $(M-uv) \oplus (M' - ux)$ and by Lemma 1, it is not an augmenting path. Thus, $P \oplus (M-uv)$ is maximal in $G-u$. Note that since M' is maximal in G and $v \in C_1(G)$, M' covers v with a thin edge. Thus, $P \oplus (M-uv)$ covers v with a thin edge. Also, $M - uv$ is maximal in $G-u$ and misses v. Thus, $v \in D_1(G-u) \cup D_{1,2}(G-u)$. But, by the parts (i) and (ii) of Lemma 3, this is impossible. Thus, M cannot contain uv where $v \in C_1(G)$. Now, suppose M contains uv where $v \in A_{1,2}(G) \cup C_{1,2}(G)$. We claim u is incident to at least two vertices in $D_1(G) \cup D_{1,2}(G)$. Suppose there is only one such vertex v'. Note that M covers v', since otherwise, $M-uv+uv'$ is maximal and misses v, which contradicts with $v \in A_{1,2}(G) \cup C_{1,2}(G)$. Since $v' \in D_1(G) \cup D_{1,2}(G)$, there exists a maximal matching M' missing v'. Then, $M \oplus M'$ contains a path starting with an edge of M at v'. Now, consider $M \oplus P$. By Lemma 1, P is not an augmenting path with respect to M and M'. Thus, the matching $M \oplus P$ is a maximal matching in G. If the path P does not cover u, then the matching $M \oplus P$ contains uv. Since $M \oplus P$ misses v', the matching $(M \oplus P) + uv' - uv$ is maximal and misses v, which contradicts with $v \in A_{1,2}(G) \cup C_{1,2}(G)$. If the path P does covers u, then, since $u \in A_1(G)$ and u is incident to only one vertex in $D_1(G) \cup D_{1,2}(G)$, it contains $uv \in M$ and $ux \in M'$ for some $x \in A_{1,2}(G) \cup C_{1,2}(G)$. (Note that M cannot contain an edge between a vertex in $A_1(G)$ and a vertex in $C_1(G)$, as we have shown above.) But then, the matching $M \oplus P$ contains ux and the matching $(M \oplus P) - ux + uv'$ is maximal in G and misses x, which contradicts with $x \in A_{1,2}(G) \cup C_{1,2}(G)$. Thus, u is incident to at least two vertices in $D_1(G) \cup D_{1,2}(G)$. Let v_1 and v_2 be two such vertices. The matching M covers v_1 and v_2, since otherwise, one can construct a maximal matching missing v. Let M_1 be a maximal matching missing v_1. Note that since $u \in A_1(G)$, the vertex u is covered in M_1 and it cannot be matched with a vertex say u' in $A_{1,2}(G) \cup C_{1,2}(G)$, since otherwise $M_1 + uv_1 - uu'$ is maximal and misses u', contradicting with $u' \in (A_{1,2}(G) \cup C_{1,2}(G))$. Let v_2 be the vertex with which u is incident in $M_1(G)$. Then, there exists a maximal matching M_2 missing v_2. Without loss of generality, we can assume that M_2 contains uv_1. (Note that $M_1 + uv_1 - uv_2$ is such a matching missing v_2.)[4] Now consider the graph $G - u$. The matchings $M - ux$, $M_1 - uv_2$ and $M_2 - uv_1$ are all maximal matchings in $G - u$ and also, both matchings $M_1 - uv_2$ and $M_2 - uv_1$ miss v_1 and v_2 in $G - u$. Now, consider $(M - ux) \oplus (M_1 - uv_2)$; it contains a path P_1 starting with an edge of M at v_1. Similarly, $(M - ux) \oplus (M_2 - uv_1)$ contains a path P_2 starting with an edge of M at v_2. Note that either these two paths do not intersect or they coincide with each other. In the latter case, there is an alternating path starting and ending with an edge of M, which, by Lemma 1, contradicts with the maximality of $M_1 - uv_2$ and $M_2 - uv_1$. Thus, the paths P_1 and P_2 do not intersect. Suppose neither of the paths end at x. Then, $(M - ux) \oplus P_1$ is a maximal matching in $G - u$ missing v_1 and $((M - ux) \oplus P_1) + uv_1$ is maximal in G missing x, which contradicts with $x \in A_{1,2}(G)$. Thus, exactly one of the paths P_1 and P_2 end at x. Suppose P_1 ends at x. Consider the matching $(M_1 - uv_2) \oplus P_1$. It is maximal in $G - u$ and misses x. Then, $((M_1 - uv_2) \oplus P_1) + uv_1$ is maximal in G and misses x, which contradicts with $x \in A_{1,2}(G)$. Now, suppose P_2 ends at x. Then, since P_1 and P_2 do not intersect, the path P_1 does not end at x. Also, the matching $(M - ux) \oplus P_1$ is maximal in $G - u$ and misses v_1. Then, $((M - ux) \oplus P_1) + uv_1$ is maximal in G missing x, which contradicts with $x \in A_{1,2}(G)$. ■

Definition 4 *A graph is called* ***hypomatchable*** *if for each $v \in V$, $G - v$ has a perfect matching.*

[4]This can be shown as follows as well: Suppose M_2 contains uy and misses v_1. Then, $M_2 + uv_1 - uy$ is maximal as well and misses v_2. Now, suppose M_2 contains uy and does not miss v_1. Then, since M_1 misses v_1, $M_1 \oplus M_2$ contains a path starting at v_1 with an edge of M_1. By Lemma 1, the matching $M_2 \oplus P$ is maximal as well. If the path P does not cover u, then the matching $(M_2 \oplus P) + uv_1 - uy$ is maximal as well and misses v_2. If the path P covers u, then since M_2 misses v_2 and M_1 contains uv_2, it should end with the edge uv_2 of M_1. Now, consider the matching $M_2 \oplus P$. It misses v_1 and contains v_2. Then, the matching $(M_2 \oplus P) + uv_1 - uv_2$ misses v_2 and contains uv_1.

Lemma 5 *Let $G - A_1(G)$ be the subgraph induced by removing the vertices in $A_1(G)$. The components of $G - A_1(G)$ in $D(G)$, if any, are hypomatchable, and each maximal matching of G contains a near-perfect matching of each such component.*

This result derives from Gallai's Lemma.[5] The proof is almost the same, but for the sake of completeness, we present it here.

Proof. By Lemma 3(iv), the subgraph $G - A_1(G)$ is such that if we remove all the vertices in $A_1(G)$ one-by-one, $D(G - A_1(G)) = D(G)$. Let M be a maximal matching in G. Let H be a component of $G - A_1(G)$ in $D(G)$. By Lemma 4, the matching $M - A_1(G)$ is maximal in $G - A_1(G)$. Now, consider the graph $G - A_1(G)$. Suppose that there are two maximal matchings M_1 and M_2 missing the vertices A and B respectively, such that $|A| < |B|$. Now, suppose that all the vertices in $B \setminus A$ are covered by M_1. Let $u \in B \setminus A$. Since M_2 misses u, the set $M_1 \oplus M_2$ contains a path P starting at u with an edge of M_1. Since both M_1 and M_2 are maximal, Lemma 1 implies that the path P is even. Also, since $|A \setminus B| < |B \setminus A|$, without loss of generality, we can assume that P does not end in A. Note that P does not contain any vertex of A. Then, by Lemma 1, $M_1 \oplus P$ is maximal and misses the vertices in $A \cup \{u\}$. Thus, there exists a maximal matching which misses A and at least one vertex of $B \setminus A$. Now define the binary relation $\sim$ as follows: $u \sim v$ if and only if $u = v$ or no maximal matching misses both u and v. Suppose $u \sim v$ and $v \sim w$. Let M_1 be a maximal matchings missing v and M_2 one missing u and w. But then, there is a maximal matching missing v and w, which is a contradiction. Thus, $u \sim w$ and $\sim$ is an equivalence relation. Now, since H is connected, any two vertices of H must be equivalent. Thus, any maximal matching misses at most one vertex of H. Also, since any vertex $u \in H$ is also in $D(G - A_1(G))$, $\nu(G - A_1(G)) = \nu(G - A_1(G) - u)$. Thus, the reduced submatching of M on H is a near-perfect matching of H. ■

6.3 The maximum number of kidney transplantations

Theorem 3 *Let $G = (V, E)$ be a graph and let $w \in \Re_+^E$ be a weight function. Then the maximum weight of a matching is equal to the minimum value of*

$$\sum_{v \in V} y_v + \sum_{S \in \mathcal{P}_{odd}(V)} z_S \lfloor \frac{1}{2} |S| \rfloor,$$

where $y \in \Re_+^E$ and $z \in \Re_+^{\mathcal{P}_{odd}(V)}$ satisfy

$$\sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{S \in \mathcal{P}_{odd}(V)} z_S \chi^{E[S]} \geq w.$$

[5]See Theorem 3.1.13 in Lovasz and Plummer [7].

Definition 5 *A collection $\mathcal{F}$ of sets is called* laminar *if, for all $S, T \in \mathcal{F}$, either $S \cap T = \emptyset$ or $S \subseteq T$.*

Theorem 4 *(Cunningham-Marsh formula) In Theorem 1, if w is integer, we can take y and z integer. We can take z moreover such that the collection $\{S \in \mathcal{P}_{odd}(V) : z_S > 0\}$ is laminar.*

Proposition 2 *Let (y, z) be a solution to the minimization problem in Theorem 1. For $k = 1, 2$, let $U_k = \{v \in V : y_v = k\}$, and $W_k = \{S \in \mathcal{P}_{odd} : z_S = k\}$. Then, (i)the set U_2 and the sets in W_2 are disjoint, (ii) for each $S \in W_1$, $S \cap U_2 = \emptyset$, (iii) for each $S \in W_2$, $|S \cap U_1| \leq 1$, (iv) if $S \in W_1$ and $T \in W_2$, then $S \cap T = \emptyset$.*

Proof. By the Cunningham-Marsh formula, for all $v \in V$, either $y_v \in \{0, 1, 2\}$ or $z_S \in \{0, 1, 2\}$ for some odd set S containing v.

(i) Let $S, T \in W_2$. Suppose $S \cap T \neq \emptyset$. If $S \cup T$ is an odd set, then

$$\lfloor \frac{|S|}{2} \rfloor + \lfloor \frac{|T|}{2} \rfloor = \frac{|S| + |T|}{2} - 1 \geq \frac{|S \cup T| + 1}{2} - 1 = \lfloor \frac{|S \cup T|}{2} \rfloor.$$

Let $z'_S = z'_T = 0$, $z'_{S \cup T} = 2$, and for each $S' \in \mathcal{P}_{odd} \setminus \{S, T, S \cup T\}$, $z'_{S'} = z_{S'}$. Since $E(S \cup T) \supseteq E(S) \cup E(T)$, z' is feasible; it also reduces the value of the objective function. If $S \cup T$ is an even set, let $j \in S \cup T$ and set $z'_{S \cup T - \{j\}} = 2$ and $y'_j = 2$. Note that (z', y') is feasible. Since

$$\frac{|S| + |T|}{2} - 1 \geq \frac{|S \cup T| + 2}{2} - 1 = \frac{|S \cup T - \{j\}| - 1}{2} + 1,$$

(z', y') does not increase the cost function. Thus, at an optimal solution, the sets in W_2 are disjoint. If for some $S \in W_2$, $v \in U_2 \cap S$, then for some $u \in S$, by letting $y_u = 2$, $z'_S = 0$ and $z'_{S \setminus \{u,v\}} = 2$, we obtain another feasible solution, which has the same value of the objective function.

(ii) Suppose there exists $S \in W_1$ such that S contains v for some $v \in U_2$. Note that the case $U_2 \supseteq S$ is not possible, since then, setting $z'_S = 0$ reduces the value of the objective function. Thus, there exists $u \in S$ such that $y_u \in \{0, 1\}$. If $y_u = 1$, then setting $z'_S = 0$, $z'_{S \setminus \{u,v\}} = 1$ and $y'_u = 2$ is feasible and does not increase the value of the objective function. Similarly, if $y_u = 0$, then setting $z'_S = 0$, $z'_{S \setminus \{u,v\}} = 1$ and $y'_u = 1$ is feasible and does not increase the value of the objective function.

(iii) Suppose $u, v \in U_1 \cap S$ for some $S \in W_2$. Then, set $z'_{S \setminus \{u,v\}} = 2$ and $y'_u = y'_v = 2$. The value of the objective function under (y', z') does not change.

(iv) Let $S \in W_1$ and $T \in W_2$. By the Cunningham-Marsh Theorem, the collection W_1, W_2 is laminar. Thus, either $S \subseteq T$ or $S \supseteq T$ or $S \cap T = \emptyset$. The first two cases contradict with the optimality: if $S \subseteq T$, then set $z'_S = 0$; if $S \supseteq T$, then set $z'_T = 1$. In either case, the value of the objective function is decreased. Thus, $S \cap T = \emptyset$. ■

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