Submission Number: ASSET2009-09-00026

OLIGOPOLISTIC EQUILIBRIUM AND BANKRUPTCY

Carmen Beviá
Universitat Autònoma de Barcelona

Luis Corchon
Universidad Carlos III

Abstract

In this paper we present a model of oligopoly and bankruptcy. We study allocations which are Bankruptcy-Free in the sense that no firm can drive to bankruptcy another firm without becoming bankrupt. We show how such allocations can be sustained as an equilibrium of a dynamic game. When there are two firms all equilibria in the dynamic game must include Bankruptcy-Free allocations only.

Submitted: June 06, 2009.

OLIGOPOLISTIC EQUILIBRIUM AND BANKRUPTCY*

Carmen Beviá

Departament d'Economia i d'Història Econòmica and CODE

Universitat Autònoma de Barcelona

Luis C. Corchón

Departamento de Economía

Universidad Carlos III

c/ Madrid 126, Getafe, Madrid 28903, Spain

lcorchon@eco.uc3m.es

First version April 30^{th}, 2008. This version April, 9^{th}, 2009.

Abstract

In this paper we present a model of oligopoly and bankruptcy. We study allocations which are Bankruptcy-Free in the sense that no firm can drive to bankruptcy another firm without becoming bankrupt. We show how such allocations can be sustained as an equilibrium of a dynamic game. When there are two firms all equilibria in the dynamic game must include Bankruptcy-Free allocations only.

*We are grateful to Adam Brandenburger, Luis Cabral, Jose Luis Ferreira, Michael Katz, Asher Wolinsky and the participants in the NASS, NYU for helpful suggestions. The first author acknowledges financial support from SEJ2006-27589-E, ECO2008-04756, 2005SGR-00454, and Barcelona GSE research network. The second author acknowledges financial support from SEJ2005-06167/ECON and wishes to express his gratitute for the hospitality of the departments of economics of Stern School of Business, NYU and The Australian School of Business, UNSW.

1. Introduction

The theory of Industrial Organization is one of the main applications of game theory. Concepts like Cournot equilibrium, Bertrand equilibrium or Stackelberg equilibrium are usually presented as an illustration of basic concepts of game theory like Nash equilibrium or subgame perfection. These concepts, though, apply strictu senso to one shot situations so a little bit of interpretation is needed because oligopolistic competition is seldom a once-and-for-all interaction among players. A possible interpretation is that the static game is repeated a *finite* number of times. If the one-shot game has a unique Nash equilibrium, it can be shown that the subgame perfect Nash equilibrium of the repeated game consists in playing the actions corresponding to the Nash equilibrium of the one-shot game each time the game is repeated.[1,2]

Unfortunately, the repetition, finite or infinite, of the static game opens new possibilities that are not considered in the equilibrium concepts mentioned above. Suppose that a firm, say 1, takes in the first period an action that produces negative profits for all other firms but positive profits for firm 1.[3] If firms do not posses liquid assets to face losses, all firms but 1 become bankrupt. And, it is not clear that these firms will ever produce again.[4] In fact it is very likely that firm 1 becomes a monopolist. This has been called *predatory pricing* and the usual explanations are based on incomplete information (Milgrom and Roberts [1982]), the learning curve (Cabral and Riordan [1994]) or that firms play a war of attrition game (Roth [1996]).

In this paper we study a model where firms may become bankrupt without resorting to any of the explanations quoted above: we will assume that information is complete, firms do not learn and they play standard price or quantity-setting games. Notice that our framework is not one of a repeated game because the short-run game changes with the strategies chosen in the past. Rather it is a special case of a stochastic game, a concept introduced by Shapley in 1953 (see Neyman and Sorin 2003) in which the transition is degenerate. Such games are usually termed Dynamic Games (Neme and Masso 1996).

[1] An alternative explanation is that firms play a supergame in which the discount rate is very close to zero.

[2] Uniqueness is essential if the game is repeated a finite number of times, see, e. g. Benoit and Krishna (1985).

[3] This may be explained because firm 1 possesses a better technology, or the demand function of this firm is above from those of any other firm, etc.

[4] For instance, Sharfstein and Bolton (1990) study the optimal design of a contract between an entrant firm and its financiers, assuming that funding terminates if profits are low.

A full dynamic analysis of bankruptcy games is complicated by two reasons: Firstly one has to consider that firms accumulate assets over time. Secondly, if the game is played an infinite number of times, the characterization of equilibrium strategies becomes difficult because moves on time can be very convoluted. Rosenthal and Rubinstein (1984) studied such a situation in the case of two players under the assumption that each player regards the ruin of the other player as the best possible outcome and her own ruin as the worst possible outcome. They characterize a subset of all Nash equilibria. Despite of the geometrical appeal of their results, it is not clear how to interpret them in an IO framework. Rosenthal and Spady (1989) consider a prisoner's dilemma repeated in continuous time between two firms where each ruined firm is immediately replaced by a new entrant. Again, it is not totally clear which kind of economic situation is captured by this model.

In this paper we approach the question of bankruptcy in oligopoly as follows:

Our first step is to define a set of actions (prices or quantities) that are Bankruptcy-Free, BF in the sequel, in the static game: This is the set of actions that yield non-negative profits to all firms and such that no firm can be driven to bankruptcy by an action of another firm which obtains non-negative profits with this action. The concept of BF captures the opportunities for ruining other players that are not captured by concepts like Cournot or Bertrand equilibria.[5]

Our second step consists in characterizing the set of BF actions under alternative assumptions. When the product is homogeneous and firms compete in quantities in the case where average cost are non-decreasing (Proposition 1) or where firms have identical decreasing average costs (Proposition 2). When the product is heterogeneous, there are two firms competing in quantities and costs are linear (Propositions 3, and 4). Finally, we consider the case of price competition with linear demand and costs (Proposition 5).

Our final step is to consider a dynamic game where BF actions emerge as an equilibrium. We first notice that if the Nash equilibrium corresponding to the static game is BF, this allocation can be supported as a Subgame Perfect Nash equilibrium (Proposition 6). For the case of two players with a few additional assumptions we provide a "Folk Theorem" showing that for a discount rate sufficiently close to one all outputs that yield profits larger than the maximin strategy restricted to the set of BF allocation, can be supported as a Nash equilibrium of the dynamic game (Proposition 8). We will also show that not all BF allocations can be supported as a Nash equilibria of the

[5] Rosenthal and Rubinstein (1984) provide necessary conditions for non-bankruptcy to be an equilibrium outcome of a dynamic game, see Sections 4 and 5, op. cit.

dynamic game by showing that, "generically", certain BF allocation cannot be supported as a Nash equilibrium (Proposition 9). Finally, we show for the duopoly case a converse of Propositions 6 and 8 showing that when the discount rate is sufficiently close to one, any Nash equilibrium must yield BF allocations (Proposition 10).

We end this Introduction by providing some remarks on our approach.

Despite the fact that we provide game-theoretical support for our notion, we do not endorse the view that this is the unique possible interpretation of why firms choose BF allocations. There may be other reasons such that to avoid bankruptcy might be of paramount value in the firm, or that managers analyze the pricing/output setting problem in terms of decision theory and they are extremely risk-averse so they would never choose an allocation in which they can be driven out of business, i.e. they play a kind of minimax strategy.

Coalitions of firms are not considered. This is irrelevant in the case of two firms. But when three or more firms are in the market our approach does not pinpoint all the bankruptcy opportunities open to firms and can only be considered as a first cut to the problem.

The option of buying a firm is not present. This can be regarded as a flaw of our model since the classical criticism of predatory pricing is that buying the opponent may be a cheaper and safer strategy than ruining it. We do not deny that buying competitors plays an important role in business practices. But we contend the view that under the option of buying the option of ruining a competitor is irrational. Firstly, buying competitors may be forbidden by a regulatory body because of its anticompetitive effects. Secondly, when the owner of a firm sell it to competitors, this does not stop her from creating a new firm and finance it with the money she received from selling the old one. In other words, selling a firm is not equivalent to a contract where the owner commits not to enter into a market again. And bankruptcy may be the only credible way of getting rid of a competitor. Finally, buying and ruining competitors may complement each other because the acquisition value may depend on the aggressiveness of the buyer in the past, see Burns (1986) for some evidence in the American tobacco industry. Thus it seems that a better understanding of the mechanism of ruin might help to further enhancement of our understanding of how the buying mechanism works in this case.

2. The Model

There are n firms. Each firm, say i, has an action space denoted by S_i. An action could be an output or a price set by a firm. An action profile is a vector of actions $s \in \Pi_{i=1}^{n} S_i$. Let $s_{-i} = (s_1, .., s_{i-1}, s_{i+1}, .., s_n)$, and $(\hat{s}_i, s_{-i}) = (s_1, .., s_{i-1}, \hat{s}_i, s_{i+1}, .., s_n)$. The profit of firm i depends on the action profile and is denoted by $\pi_i(s)$.

Definition 1. *An action profile $\hat{s} = (\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$ is Bankruptcy-Free (BF) if:*

a) $\pi_i(\hat{s}) \geq 0$, for all $i \in \{1, .., n\}$.

b) For all s_j such that $\pi_j(\hat{s}_{-j}, s_j) \geq 0$, $\pi_i(\hat{s}_{-j}, s_j) \geq 0$ for all $i \neq j$.

In words, a profile of actions is BF if it yields non negative profits to all firms and no firm can change its action, obtain non-negative profits and ruin other player.

In what follows we consider two scenarios: one with quantity setter firms, and the other with price setter firms.

In the quantity setter case, $s_i = x_i$ where $x_i \in \mathbb{R}_+$ denotes the quantity set by firm i.

In the price setter case, $s_i = p_i$ where $p_i \in \mathbb{R}_+$ denotes the price set by firm i.

We start analyzing the existence of Bankruptcy-free actions in the case of homogeneous product and quantity setter firms.

3. Quantity Setter Firms. Homogeneous Product

Given a quantity profile $x = (x_1, .., x_n)$, let $p(\sum_{i=1}^{n} x_i)$ be the inverse demand function, we assume that p is strictly decreasing. Let $c_i(x_i)$ be the cost of producing x_i for firm i. Profits for firm i given profile x are given by $\pi_i(x) = p(\sum_{i=1}^{n} x_i)x_i - c_i(x_i)$. We assume that profits are concave functions. For each firm i, average cost of producing x_i is denoted by $AVC_i(x_i)$.

If a firm is producing a positive quantity we call it an active firm. Otherwise it is an inactive firm. Clearly, an action profile with all inactive firms is BF. In what follows we concentrate in the characterization of BF action profiles with at least an active firm.

We start characterizing the set of BF action profiles for n firms with non-decreasing average costs. We assume that for all firm i, and for all x_{-i} , there exist $\bar{x}_i \neq 0$ such that $\pi_i(x_{-i}, \bar{x}_i) = 0$.

Proposition 1. *Let n firms with non-decreasing average costs. An action profile $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ is BF if and only if $\pi_i(\hat{x}) \geq 0$ for all $i \in N$, and either*
(i) All firms have the same average cost, that is, $AVC_j(\hat{x}_j) = AVC_k(\hat{x}_k)$ for all j, k, or
(ii) For all active firms j, k if $AVC_j(\hat{x}_j) < AVC_k(\hat{x}_k)$, firm j can always increases its production in a way that match the average cost of firm k, keeping non negative profits. That is, there is $\tilde{x}_j$ such that $AVC_j(\tilde{x}_j) = AVC_k(\hat{x}_k)$, and $p(\sum_{i \neq j} \hat{x}_i + \tilde{x}_j) - AVC_j(\tilde{x}_j) \geq 0$.
If a firm j is inactive, then for all $x_j \neq 0$ such that $\pi_j(\hat{x}_{-j}, x_j) = 0$, $AVC_j(x_j) > AVC_k(\hat{x}_k)$ for all active firm k.

Proof. Its is obvious that the action profiles described in (i) are BF. For the action profiles described in (ii), since average cost is non-decreasing it is obvious that no firm with positive production can drive a firm with lower average cost to bankruptcy. It is also obvious that non inactive firm can enter the market and drive the active firms to bankruptcy. Let us see that neither it is possible for a firm with positive production to drive a firm with higher average cost to bankruptcy. Let $\bar{x}_j$ be such that

$$p(\sum_{i \neq j} \hat{x}_i + \bar{x}_j) \geq AVC_j(\bar{x}_j). \tag{3.1}$$

If $AVC_j(\bar{x}_j) \geq AVC_k(\hat{x}_k)$, since average cost is non-decreasing, then $p(\sum_{i \neq j} \hat{x}_i + \bar{x}_j) - AVC_k(\hat{x}_k) \geq 0$. If $AVC_j(\bar{x}_j) < AVC_k(\hat{x}_k)$, then $AVC_j(\bar{x}_j) < AVC_j(\tilde{x}_j)$, and since average cost is non-decreasing, $\bar{x}_j < \tilde{x}_j$. Then $p(\sum_{i \neq j} \hat{x}_i + \bar{x}_j) > p(\sum_{i \neq j} \tilde{x}_i + \tilde{x}_j) \geq AVC_j(\tilde{x}_j) = AVC_k(\hat{x}_k)$. Therefore, for all $k \neq j$, with higher average cost $p(\sum_{i \neq j} \hat{x}_i + \bar{x}_j) - AVC_k(\hat{x}_k) \geq 0$.
Finally, let us see that any other action profile can not be BF.
Let $x = (x_1, ..., x_n)$ be such that $\pi_i(x) \geq 0$ for all $i \in N$, and suppose that there are two active firms, j and k, with $AVC_j(x_j) < AVC_k(x_k)$, and such that, for $\tilde{x}_j$ with $AVC_j(\tilde{x}_j) = AVC_k(x_k)$, $p(\sum_{i \neq j} x_i + \tilde{x}_j) - AVC_j(\tilde{x}_j) < 0$. Since $\pi_j(x) > 0$ and the price-average cost difference is decreasing, firm j can decrease production and make cero profits. That is, there is $\bar{x}_j < \tilde{x}_j$ such that $p(\sum_{i \neq j} x_i + \bar{x}_j) - AVC_j(\bar{x}_j) = 0$. Since $AVC_j(\bar{x}_j) < AVC_j(\tilde{x}_j) = AVC_k(x_k)$, $p(\sum_{i \neq j} x_i + \bar{x}_j) - AVC_j(\bar{x}_j) > p(\sum_{i \neq j} x_i + \bar{x}_j) - AVC_k(x_k)$, which implies that firm j can drive firm k to bankruptcy.
Let $x = (x_1, ..., x_n)$ be such that $\pi_i(x) \geq 0$ for all $i \in N$, and suppose that there are two active firms, j and k, with $AVC_j(x_j) < AVC_k(x_k)$, and such that firm j can never match the average cost of firm k, that is, for all $\tilde{x}_j$, $AVC_j(\tilde{x}_j) < AVC_k(x_k)$. Let $\bar{x}_j$ be such that $p(\sum_{i \neq j} x_i + \bar{x}_j) - AVC_j(\bar{x}_j) = 0$.

By our assumptions $\bar{x}_j$ exist, and since $AVC_j(\bar{x}_j) < AVC_k(x_k)$, firm k is bankrupt.
Finally, if for an inactive firm k there is an $\bar{x}_k \neq 0$ such that $AVC_j(x_j) > AVC_k(\bar{x}_k)$ and $p(\sum_{i \neq k} x_i + \bar{x}_k) = AVC_j(x_j)$, firm k can increase his production above $\bar{x}_k$ and make firm j bankrupt keeping positive profits. ■

Proposition 1 implies that if firms have identical constant average cost, all quantity profiles that yield non-negative profits for all firms are BF. Also from Proposition 1 it is easy to characterize the set of BF action profiles for two firms with identical non-decreasing average cost. We give the details in the following Corollary.

Corollary 1. *Let two firms with increasing average cost. Let* $(\hat{x}_1, \hat{x}_2)$ *be such that for all* $i \in \{1, 2\}$, $\pi_i(\hat{x}_1, \hat{x}_2) = 0$ *and* $\hat{x}_i \neq 0$, *A quantity profile* (x_1, x_2) *is BF if and only if* $x_i \leq \hat{x}_i$ *for all* $i \in \{1, 2\}$ *and* $\pi_i(x_1, x_2) \geq 0$.

Proof. For the if part, notice first that by Proposition 1 $(\hat{x}_1, \hat{x}_2)$ is BF. Also, (x_1, x_2) is BF if the average cost for both firms are the same. In any other case, suppose without lost of generality that $AVC_i(x_1) < AVC_j(x_2)$.

Let $\bar{x}_1$ be such that $AVC_i(\bar{x}_1) = AVC_j(x_2)$, and suppose that $\pi_1(\bar{x}_1, x_2) < 0$. Since average cost is increasing and $x_2 \leq \hat{x}_2$, $AVC(\bar{x}_1) \leq AVC(\hat{x}_2)$, therefore $\bar{x}_1 \leq \hat{x}_2 = \hat{x}_1$. Since $p(\bar{x}_1 + x_2) < AVC(\bar{x}_1)$, $p(\bar{x}_1 + x_2) < AVC(\hat{x}_1)$. Thus, $p(\bar{x}_1 + x_2)\hat{x}_1 - c(\hat{x}_1) < 0$. And since $\bar{x}_1 \leq \hat{x}_1$ and $x_2 \leq \hat{x}_2$,

$$p(\hat{x}_1 + \hat{x}_2)\hat{x}_1 - c(\hat{x}_1) \leq p(\bar{x}_1 + x_2)\hat{x}_1 - c(\hat{x}_1) < 0, \tag{3.2}$$

which it is not possible since $p(\hat{x}_1 + \hat{x}_2)\hat{x}_1 - c(\hat{x}_1) = 0$. Thus, $\pi_1(\bar{x}_1, x_2) \geq 0$, and therefore, by Proposition 1 (x_1, x_2) is BF.
For the only if part we have to see that any other possible action profile different from the one described in the Corollary can not be BF. Let (x_1, x_2) be such that $\pi_1(x_1, x_2) \geq 0$ and $x_1 > \hat{x}_1$. First of all notice that $x_1 + x_2 \leq \hat{x}_1 + \hat{x}_2$, otherwise firm 1 will have negative profits at (x_1, x_2). Thus $x_2 < \hat{x}_2$. Let us see that firm 2 can make firm 1 bankrupt. Let $\bar{x}_2$ be such that $p(x_1 + \bar{x}_2) = AVC(\bar{x}_2)$. Notice that $\bar{x}_2 > x_2$. Furthermore $\bar{x}_2 < \hat{x}_2$, otherwise $x_1 + \bar{x}_2 > \hat{x}_1 + \hat{x}_2$ which will imply negative profits for both firms. Thus $\bar{x}_2 < \hat{x}_2 = \hat{x}_1 < x_1$ and $AVC(\bar{x}_2) < AVC(x_1)$ which implies negative profits for firm 1. ■

Also from Proposition 1, we can deduce that, for different constant average cost, only the firm with the smallest average cost can be active.

Corollary 2. *Let n firms with constant average cost, c_i, such that $c_1 < c_2 < .... < c_n$. An action profile with active firms $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ is BF if and only if $\hat{x}_1 > 0$, $\hat{x}_j = 0$ for all $j \in \{2, .., n\}$ and $\pi_1(\hat{x}) \geq 0$.*

Proof. Clearly, $(\hat{x}_1, 0, 0, ..., 0)$ is BF. Let us see that no other action profile with active firms is BF. By Proposition 1, under different constant average cost, only profiles with just one active can be BF. The active firm has to be firm one. Otherwise firm one can enter the market, produce $x_1' > 0$ such that $p(\sum_{j \neq 1} x_j + x_1') = c_2 - \varepsilon$ with $0 < \varepsilon < c_2 - c_1$, and make the active firm bankrupt. Notice that x_1' always exist because since $p()$ is strictly decreasing,

$$x_1' = p^{-1}(c_2 - \varepsilon) - \sum_{j \neq 1} x_j, \tag{3.3}$$

and since the active firm, let us say firm j, has positive profits,

$$p(\sum_{j \neq 1} x_j) = p(\sum_{j} x_j) \geq c_j > c_2 > c_2 - \varepsilon, \tag{3.4}$$

$x_1' \geq 0$. With this action, firm 1 can enter the market, can have positive profits, and it makes firm j bankrupt. ■

To close the analysis with homogeneous product, we characterized the BF action profiles with decreasing average cost.

Proposition 2. *Let n firms with identical decreasing average cost, and suppose that there is x such that $p(x) = 0$. The BF action profiles are:*
(i) The action profile $(\hat{x}_1, .., \hat{x}_n)$ such that: $\hat{x}_i = \hat{x}_j \neq 0$ for all $i, j \in \{1, ..., n\}$, $\pi_i(\hat{x}_1, .., \hat{x}_n) = 0$ for all $i \in \{1, ..., n\}$, and $|(AVC(\hat{x}_i))'| < \left|\frac{\partial p(\hat{x}_1 + .. + \hat{x}_n)}{\partial x_i}\right|$.
(ii) The action profiles $(\bar{x}_1, ..., \bar{x}_n)$ such that there is a unique active firm. For the active firm i, $\pi_i(\bar{x}_1, .., \bar{x}_n) \geq 0$ and $\bar{x}_i \geq \sum_{k \neq j} \hat{x}_k$ for all $j \neq i$, where $(\hat{x}_1, .., \hat{x}_n)$ is described in (i).

Proof. Step 1. Let us see first that no action profile $(x_1, ..., x_n)$ such that at least two firms are active and at least one has strictly positive profits is BF.
Without lost of generality, let us suppose that $\pi_1(x_1, .., x_n) > 0$, which implies that $x_1 > x_2 > 0$. Let y be such that $\pi_1(y, x_2, .., x_n) = 0$, let us see that y exist and is such that $y > x_1$. Since $\pi_1(x_1, .., x_n) > 0$, $p(x_1 + .. + x_n) > 0$. By the assumption on the inverse demand function there is z such that $p(z + x_2 + .. + x_n) = 0$, and since p is decreasing, $z > x_1$. Therefore, $\pi_1(z, x_2) < 0$. By

continuity of π_1 there is $y \in (x_1, z)$ such that $\pi_1(y, x_2, .., x_n) = 0$. If firm 1 increases its production from x_1 to y, price will be equal to the average cost of y, and since average cost is decreasing, $AVC(y) < AVC(x_1) < AVC(x_2)$. Thus, firm 2 is bankrupt.

Step 1 tell us that only action profiles such that all firms have zero profits or action profiles where only one firm has positive profits and all others are not active can be BF.

Step 2. Let us see that an action profile $(x_1, ..., x_n)$ such that for all $i \in \{1, ..., n\}$, $\pi_i(x_1, .., x_n) = 0$, but $|(AVC(x_i))'| \geq \left|\frac{\partial p(x_1+..+x_n)}{\partial x_i}\right|$ for some agent i is not BF.

Let us suppose that the firm with the above characteristics is firm 1. Since $|(AVC(\hat{x}_1))'| \geq \left|\frac{\partial p(x_1+..+x_n)}{\partial x_1}\right|$, firm 1 can increases a little bit its output and get non negative profits. Since price will decrease, all other firms will be bankrupt.

Step 3. If $(\hat{x}_1, ..., \hat{x}_n)$ is such that for all $i \in \{1, ..., n\}$, $\pi_i(\hat{x}_1, .., \hat{x}_n) = 0$, and $|(AVC(\hat{x}_i))'| < \left|\frac{\partial p(\hat{x}_1+..+\hat{x}_n)}{\partial x_i}\right|$, it is easy to see that all firms get negative profits by increasing the output. Thus, those action profiles are BF.

Step 4. Let us see that an action profile $(x_1, ..., x_n)$ such that there is a unique active firm $i \in \{1, ..., n\}$ with $\pi_i(x_1, .., x_n) \geq 0$, and $x_i < \hat{x}_i$ is not BF.

Suppose that the inactive firm j decides to enter the market and produces $x_j^* \neq 0$, such that $x_j^* + x_i = \hat{x}_1 + .. + \hat{x}_n$. Since $x_i < \hat{x}_i$, $x_j^* > \hat{x}_j$. Since average cost is decreasing $AVC(x_j^*) < AVC(\hat{x}_j)$, and thus $\pi_j(0, .., x_i, 0, .., x_j^*) > 0$. Since $p(x_j^* + x_i) = p(\hat{x}_1 + .. + \hat{x}_n)$, and $x_i < \hat{x}_i$, $p(x_j^* + x_i) - AVC(x_i) < p(\hat{x}_1 + .. + \hat{x}_n) - AVC(\hat{x}_i) = 0$, which implies that agent i is bankrupt.

Step 5. Finally, let us see that the action profiles describe in (ii) are BF.

For that, it is enough to show that for all $j \neq i$, and for all x_j^* such that $\pi_j(x_j^*, \bar{x}_{-j}) \geq 0$, $\pi_i(x_j^*, \bar{x}_{-j}) \geq 0$. First let us see that $x_j^* \leq \hat{x}_j$. Suppose that $x_j^* > \hat{x}_j$, since $\bar{x}_i \geq \sum_{k \neq j} \hat{x}_k$, $\bar{x}_i + x_j^* > \sum_{k=1}^n \hat{x}_k$, but by the definition of $(\hat{x}_1, .., \hat{x}_n)$ and given that profits are concave, $\pi_j(x_j^*, \bar{x}_{-j})$ will be negative. Thus $x_j^* \leq \hat{x}_j$. Therefore $AVC(x_j^*) \geq AVC(\hat{x}_j) = p(\hat{x}_1 + .. + \hat{x}_n)$ which implies that $p(\hat{x}_1 + .. + \hat{x}_n) \leq p(\bar{x}_i + x_j^*)$, and given that $\bar{x}_i \geq \hat{x}_i$, $p(\bar{x}_i + x_j^*) - AVC(\bar{x}_i) \geq p(\hat{x}_1 + .. + \hat{x}_n) - AVC(\hat{x}_i) = 0$. ■

In the following example we illustrate graphically the set of BF action profiles for two firms and a linear demand function.

Example 1. *Let us consider two firms with cost functions* $c_i(x_i) = k + cx_i$, *and let* $p(x_1 + x_2) =$

$a - x_1 - x_2$, $a > c$. Let $\alpha = a - c$. Let us see the area where both firms has positive profits, that is

$$p(x_1 + x_2) \geq \frac{k}{x_i} + c, \text{ for } i = 1, 2. \tag{3.5}$$

or equivalently,

$$x_2 \leq \alpha - x_1 - \frac{k}{x_1}, \text{ and} \tag{3.6}$$

$$x_1 \leq \alpha - x_2 - \frac{k}{x_2}. \tag{3.7}$$

Figure 1 illustrates this situation. It is easy to see in Figure 1 the action profiles that are not BF. Point $B = (B_1, B_2)$ is a BF action profile such that $\pi_i(\hat{x}_1, \hat{x}_2) = 0$ for both firms and $|(AVC(\hat{x}_i))'|$ $< \left|\frac{\partial p(\hat{x}_1, \hat{x}_2)}{\partial x_1}\right|$. Notice that there is another action profile in this example where both agents have zero profits, but this one is not BF because it does not satisfy the above requirement. In that profile, firm 1 can increase production, and since his average cost at that point decreases more than what the price decreases at that point, firm 1 will get positive profits and will make firm 2 bankrupt.

We can also see in the example BF action profiles of the second type described in Proposition 2. All action profiles such that $x_2 = 0$, $x_1 \geq B_1$ and $\pi_1(x_1, 0) \geq 0$, or $x_1 = 0$, $x_2 \geq B_2$ and $\pi_2(0, x_2) \geq 0$ are BF. It is easy to see in Figure 1 why any other action profile can not be BF.

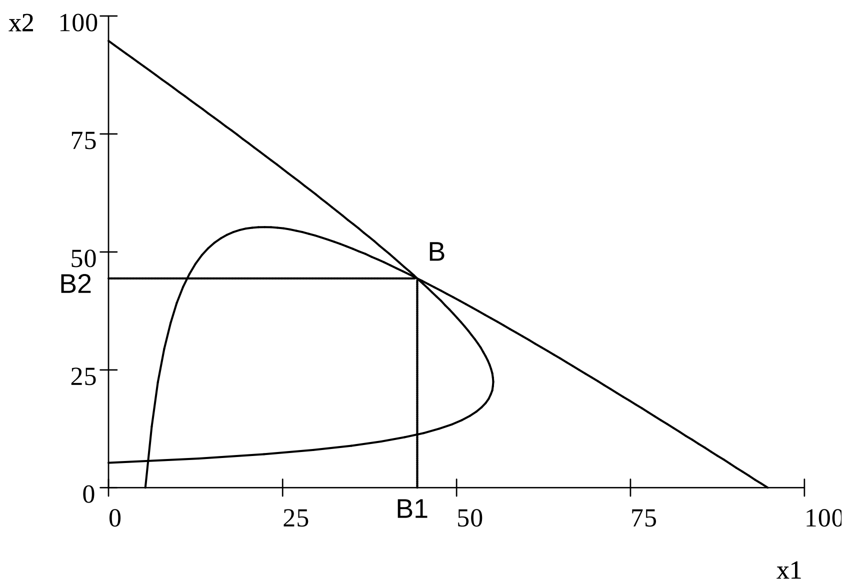

Figure 1

4. Quantity Setter Firms. Heterogeneous Product

We explore in this section the existence of $BF's$ action profiles when goods are heterogeneous and firms are quantity setters.

We consider two firms $i \in \{1,2\}$, with constant average cost c_i. Each firm faces an inverse demand function such that for each $(x_1, x_2) \in \Omega$, where Ω is a close rectangular region in $\mathbb{R}^2_+$, $p_i = p_i(x_1, x_2)$ with

$$\frac{\partial p_i(x_1,x_2)}{\partial x_j} < 0, \;\; j \in \{1,2\} \text{ and } \left|\frac{\partial p_i(x_1,x_2)}{\partial x_i}\right| > \left|\frac{\partial p_i(x_1,x_2)}{\partial x_j}\right|. \tag{4.1}$$

The vector (x_1, x_2) give non negative profits for both firms if and only if:

(i) $p_i(x_1, x_2) - c_i \geq 0$, and $x_i \geq 0$ for all i, or

(ii) $p_1(x_1, x_2) - c_1 \geq 0$, $x_1 \geq 0$, and $x_2 = 0$, or

(iii) $p_2(x_1, x_2) - c_2 \geq 0$, $x_2 \geq 0$, and $x_1 = 0$, or

(iv) $x_1 = x_2 = 0$.

Let $F : \Omega \to \mathbb{R}^2$ be such that $F(x_1, x_2) = (c_1 - p_1(x_1, x_2), c_2 - p_2(x_1, x_2))$.
By the assumptions on the inverse demand functions (4.1), the Jacobian matrix of F is dominant diagonal and has positive diagonal entries, thus the Jacobian is a P-matrix. Then F is univalent in Ω (see Gale and Nikaido (1965)). Thus, if there is an $(x_1, x_2) \in \Omega$ such that $F(x_1, x_2) = (0, 0)$, (x_1, x_2) is unique. Whenever such vector exist, we will denote it by (x_1^0, x_2^0).
The properties of F imply that the system

$$p_1(x_1, x_2) - c_1 = 0 \tag{4.2}$$

$$p_2(x_1, x_2) - c_2 = 0 \tag{4.3}$$

either has no solution in Ω or it has a unique solution $(x_1^0, x_2^0) \in \Omega$. The characterization of the BF's action profiles depends heavily on this fact.

Before presenting the main result, we illustrate in the following example the BF's profiles. The example gives us all the intuitions of the general case.

Example 2. *Suppose that the inverse demand functions are* $p_i = \alpha_i - b_i x_i - d_i x_j$, *with* $\alpha_i > c_i$, *and* $b_i > d_i > 0$ *for all* i. *Let* $a_i = \alpha_i - c_i$. *In the following pictures we show the area where profits are non negative for each firm (the whole area enclosed by the triangle plus the corresponding axis,* *all the points in the domain with $x_1 = 0$ for firm 1, and all the points in the domain with $x_2 = 0$ for firm 2).*

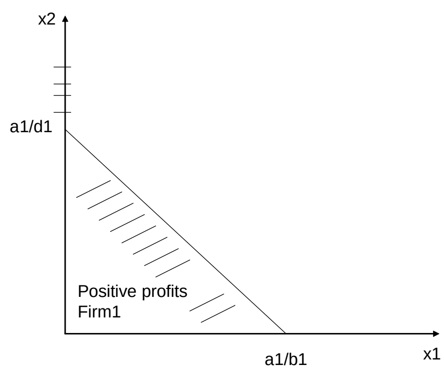

Figure 2

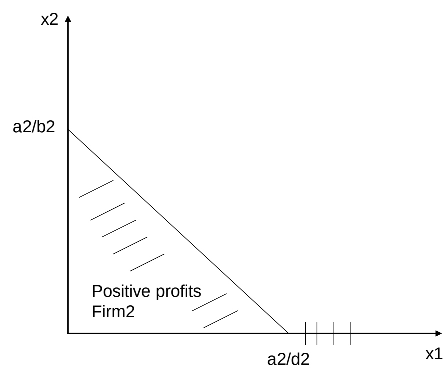

Figure 3

Two main possible cases can be generated here depending on the relation between the two areas above.

(i) The two straight lines do not cross. That is, either $a_1/d_1 < a_2/b_2$ and $a_1/b_1 < a_2/d_2$, or $a_1/d_1 > a_2/b_2$ and $a_1/b_1 > a_2/d_2$. Let us analyze the first case, the second one is identical but with the roles of firm 1 and 2 interchanged.

Notice that only the profiles such that $\hat{x}_1 = 0$, $\hat{x}_2 \geq 0$ and $\pi_2(0, \hat{x}_2) \geq 0$ are BF (the segment in the vertical axis between zero an a_2/b_2 in Figure 4). The vectors that give non negative profits for both firms are those in the area enclosed by the small triangle and the vertical axis up to a_2/b_2. Given any of those vectors, it is impossible for firm 1 to drive firm 2 to bankruptcy keeping non negative profits. Whenever firm 1 has non-negative profits by increasing its production firm 2 also has non-negative profits. However, for any other point in the area such that $x_1 > 0$, firm 2 can increases its production, keeping positive profits and make firm 1 bankrupt (for example point x in Figure 4).

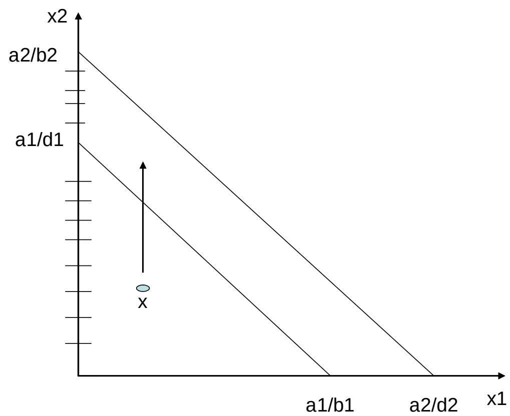

Figure 4

(ii) The two straight lines cross. Given the assumptions on the demands, there is only one possibility here: $a_1/d_1 > a_2/b_2$, $a_1/b_1 < a_2/d_2$ *(see Figure 5).*

Only the following action profiles are BF*:*

(a) The vector (x_1^0, x_2^0) *(point F in Figure 5).*

(b) All the profiles $(\hat{x}_1, \hat{x}_2)$ *such that* $0 \leq \hat{x}_1 \leq x_1^0$, $0 \leq \hat{x}_2 \leq x_2^0$ *and* $\pi_i(\hat{x}_1, \hat{x}_2) \geq 0$ *(the square in Figure 5).*

No firm can make the other one bankrupt for any point in the square. However, for points like y *in Figure 5, firm 1 can make firm two bankrupt by increasing output. For points like* x *in Figure 5, is firm 2 the one that can make firm 1 bankrupt.*

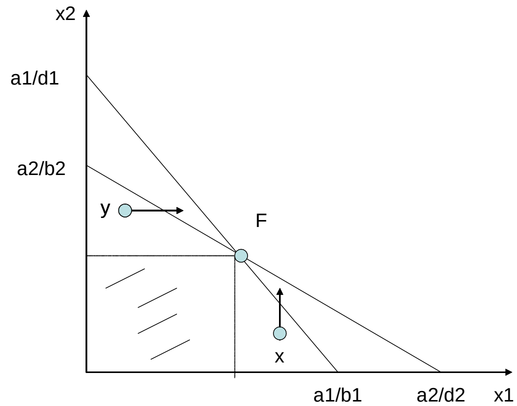

Figure 5

In the following propositions we formalize the findings of the above example for general inverse demands functions with the characteristics described in (4.1).

We introduce first some useful notation and a Lemma.

Let $S_i = \{(x_1, x_2)/p_i(x_1, x_2) - c_i \geq 0\}$. First notice that since the inverse demand function satisfies (4.1), $S_i \neq S_j$.

We start by Proposition 3, which correspond to the generalization of case (i) in the example.

Proposition 3. *If $S_i \subset S_j$, an action profile $(\hat{x}_1, \hat{x}_2)$ is BF if and only if $\hat{x}_i = 0$, $\hat{x}_j \geq 0$ and $\pi_j(\hat{x}_1, \hat{x}_2) \geq 0$.*

Proof. Suppose, without lost of generality, that $S_1 \subset S_{2.}$
Clearly, since $\hat{x}_1 = 0$, firm 2 can not do anything to drive firm 1 to bankruptcy. Neither can firm 1 by producing a positive quantity, since whenever it has positive profits, firm 2 also has it. Let us see that no any other action profile can be BF (see Figure 4). Let (x_1, x_2) be such that $x_1 > 0$ and $\pi_i(x_1, x_2) \geq 0$ for $i \in \{1, 2\}$. Since there is no $(x_1, x_2) \in \Omega$ with $F(x_1, x_2) = (0, 0)$, and $S_1 \subset S_2$, an x_2' such that $p_1(x_1, x_2') - c_1 = 0$, implies $p_2(x_1, x_2') - c_2 > 0$. Thus, firm 2 can increase its output slightly above x_2' and make firm 1 bankrupt. ■
Now let us suppose that (x_1^0, x_2^0) such that $F(x_1^0, x_2^0) = (0, 0)$ exist.
Let $S_i^- = \{(x_1, x_2)/x_1 \leq x_1^0, p_i(x_1, x_2) - c_i \geq 0\}$, and $S_i^+ = \{(x_1, x_2)/x_1 > x_1^0, p_i(x_1, x_2) - c_i \geq 0\}$.
In Figure 5 S_1^- is the set of points in the area limited by $(0, a_1/d_1)$, $F = (F_1, F_2)$, $(F_1, 0)$,and $(0, 0)$;

S_2^- is the set of points in the area limited by $(0, a_2/b_2)$, $F = (F_1, F_2)$, $(F_1, 0)$, $(0, 0)$; S_1^+ is the set of point in the area limited by $F = (F_1, F_2)$, $(F_1, 0)$, $(a_1/b_1, 0)$; and finally S_2^+ is is the set of point in the area limited by $F = (F_1, F_2)$, $(F_1, 0)$, $(a_2/d_2, 0)$

The following Lemma is useful to formalize the result illustrated in Figure 5.

Lemma 1. *$S_2^- \subset S_1^-$ and $S_1^+ \subset S_2^+$.*

Proof. Since $\frac{\partial p_i(x_1^0, x_2^0)}{\partial x_j} < 0$ for all $i, j \in \{1, 2\}$, by the implicit function theorem $p_1(x_1, x_2) - c_1 = 0$ defines x_2 as a function of x_1 in a neighborhood of (x_1^0, x_2^0), that is, $x_2 = f_1(x_1)$ and by the properties of the inverse demand function (4.1), $\left|f_1'(x_2^0)\right| > 1$. Also, by the implicit function theorem $p_2(x_1, x_2) - c_2 = 0$ defines x_2 as a function of x_1 in a neighborhood of (x_1^0, x_2^0), that is, $x_2 = f_2(x_1)$ and by the properties of the inverse demand function (4.1), $\left|f_2'(x_2^0)\right| < 1$. Thus $S_2^- \subset S_1^-$ and $S_1^+ \subset S_2^+$. ■

Finally, Proposition 4 formalizes case (ii) in the example.

Proposition 4. *Suppose that inverse demand functions are such (x_1^0, x_2^0) exist. An action profile, $(\hat{x}_1, \hat{x}_2)$, is BF if and only if $0 \leq \hat{x}_1 \leq x_1^0$, $0 \leq \hat{x}_2 \leq x_2^0$ and $\pi_i(\hat{x}_1, \hat{x}_2) \geq 0$ for all $i \in \{1, 2\}$.*

Proof. Let $(\hat{x}_1, \hat{x}_2)$ be such that $0 \leq \hat{x}_1 \leq x_1^0$, $0 \leq \hat{x}_2 \leq x_2^0$ and $\pi_i(\hat{x}_1, \hat{x}_2) \geq 0$ for all $i \in \{1, 2\}$. Since $S_2^- \subset S_1^-$ there is no way firm 2 can make firm 1 bankrupt keeping positive profits. A similar argument works for firm 1 since $S_1^+ \subset S_2^+$.

Let us see that no other action profile can be BF.

Let $(\bar{x}_1, \bar{x}_2)$ be a action profile that does not satisfy the properties described in the proposition. Then, $(\bar{x}_1, \bar{x}_2)$ has to be of type x or y represented in Figure 5.

Suppose $(\bar{x}_1, \bar{x}_2)$ is of type x, that is, $\bar{x}_1 > x_1^0$, $\bar{x}_2 \leq x_2^0$ and $\pi_i(\bar{x}_1, \bar{x}_2) \geq 0$ for $i \in \{1, 2\}$. Let $x_2' \geq \bar{x}_2$ be such that $\pi_1(\bar{x}_1, x_2') = 0$. Since $S_1^+ \subset S_2^+$, $\pi_2(\bar{x}_1, x_2') > 0$. Thus firm 2 can drive firm 1 to bankruptcy by slightly increasing its production above x_2'.

Suppose $(\bar{x}_1, \bar{x}_2)$ is of type y, that is, $\bar{x}_2 > x_2^0$, $\bar{x}_1 \leq x_i^0$ and $\pi_i(\bar{x}_1, \bar{x}_2) \geq 0$ for $i \in \{1, 2\}$. Let $x_1' \geq \bar{x}_1$ be such that $\pi_2(x_1', \bar{x}_2) = 0$.Since $S_2^- \subset S_1^-$, $\pi_1(x_1', \bar{x}_2) > 0$. Thus, firm 1 can drive firm 2 to bankruptcy by slightly increasing its production above x_1'. ■

5. Price-Setting Firms

There are two firms. The demand function of firm i is given by $x_i = \max\{0, a_i - b_i p_i + d_i p_j\}$, $i, j \in \{1, 2\}$, with all parameters positive and $b_i > d_i$. The cost function of firm i is $C_i(0) = 0$, and for $x_i \neq 0$, $C_i(x_i) = K + c_i x_i$ with $c_i < a_i$. Suppose that K is positive but arbitrarily small. Then, firm i has non negative profits iff the following two conditions are met: $p_i > c_i$ and $x_i > 0$, or equivalently, $c_i < p_i < \frac{a_i + d_i p_j}{b_i}$.

Notice that if $K = 0$ any firm by pricing above or equal to marginal cost can secure non-negative profits so in this case the set of BF allocations is very large and this concept not very interesting.

In Figure 6 below, we picture all these conditions for the case where $p_1 < 1+0.5p_2$, $p_2 < 1+0.5p_1$, $c_1 = 0.5$, $c_2 = 0.2$. All these functions are pictured with a solid line. Marginal costs are the vertical and the horizontal line. The increasing line above is the frontier of positive output for firm 1. The increasing line below is the frontier of positive output for firm 2. The area where profits are non-negative for both firms is the parallelepiped enclosed inside the four solid lines. The dotted lines represent the functions $p_1 = 1.1$ and $p_2 = 1.25$.

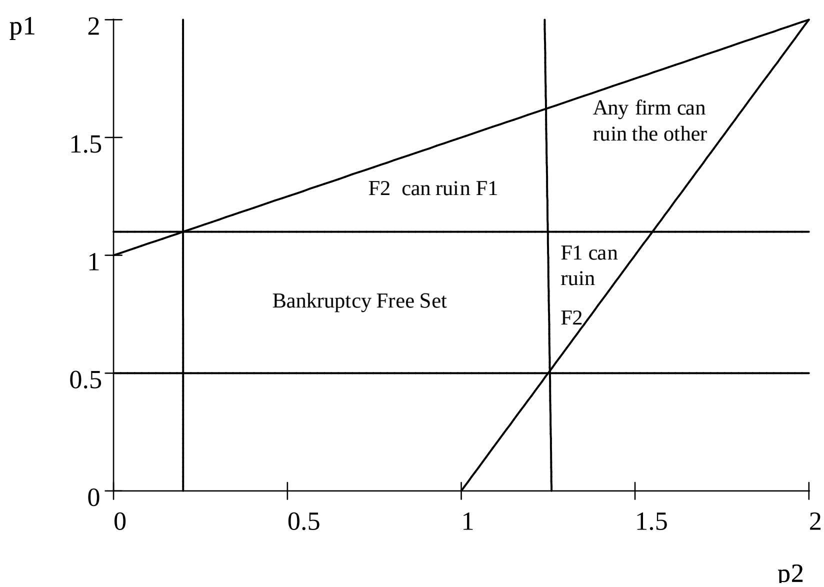

Figure 6

Now we have the following:

Proposition 5. *In Figure 6, BF allocations are those enclosed in the square delimited above and to the right by the dotted lines and below and to the left by the solid lines.*

Proof. Take a point in the upper triangle delimited by a solid line above and two dotted lines right and below. Clearly, firm 2 can decrease p_2, say very close to c_2, and make firm 1 bankrupt. In the lower triangle, delimited by the dotted lines to the left and above and by a solid line, firm 1 can decrease its price, say very close to c_1 and bankrupt firm 2. In the parallelepiped delimited by the doted lines below and to the left and by the solid lines above and to the right, both firms can bankrupt each other. This leaves the square defined in the Proposition as the only alternative. Indeed it is easily seen that no firm can bankrupt each other firm in this area. ■

6. Long-Run Competition with Bankruptcy

In this section we consider a dynamic game with an infinite horizon in which firms can be bankrupt. We will identify conditions under which such dynamic competition leads to BF allocations as defined in the previous section. Each firm, say i, has an action space denoted by S_i which can be interpreted as the output, price, etc. set by this firm. In each period, say t, each firm, say i, chooses an action s_i^t.[6] The payoffs obtained by firm i in period t are denoted by $\pi_i(s^t)$ where $s^t \in \times_{i=1}^n S_i \equiv S$ is the tuple of actions played in period $t = 1, 2, ..., \tau, ...$ Firms can not accumulate so they become bankrupt as long as they have negative profits in a period. If a firm disappears from the game in subsequent periods this firm is supposed to take an action $\bar{s}_i$ which corresponds to no action (i.e. zero output or a price for which demand is always zero). Formally, if $\pi_i(s^t) < 0$, $\pi_i(s^{t+r}) = 0$, and $s_i^{t+r} = \bar{s}_i$ for all $r = 1, 2, ..., .$ Let $\delta \in [0, 1]$ be the common discount rate. Payoffs for the game for firm i are $P_i = \sum_{t=0}^{\infty} \delta^t \pi_i(s^t)$.

For our first two propositions we will need that the set of BF actions and the Nash equilibrium of the static game have a non-empty intersection. We will show that this happens in several scenarios. In order to do that we will use the following assumptions. The first three assumptions refer to the demand.

Assumption 1. *The product is homogeneous with inverse demand function* $p = \max\{0, a - b\sum_{i=1}^n x_i\}$.

Assumption 1'. *The product is heterogeneous with inverse demand functions* $p_i = \max\{0, a_i -$

[6] Another possibility is to consider a dynamic model in which firms take actions in turn, see Cyert and de Groot (1970), Maskin and Tirole (1987), (1988a), (1988b). It can be shown that our main results also hold in this framework.

$b_i x_i - d_i x_j\}$, $b_i > d_i \geq 0$.

Assumption 1". *The product is heterogeneous with demand functions* $x_i = \max\{0, a - b_i p_i + d_i p_j\}$, $b_i > d_i \geq 0$.

Our three next assumptions refer to technology and the number of firms.

Assumption 2. *Marginal costs are constant such that* $c_1 < c_2 \leq .... \leq c_n$.

Assumption 2'. *There are two firms with identical costs functions* $c_i(x_i) = k + c x_i$ *if* $x_i > 0$ *and* $c_i(0) = 0$.

Assumption 2". *There are two firms with constant and identical marginal costs, denoted by c.*

Lemma 2. *Under any of the following conditions:*

a) Assumptions 1-2 and $a \leq 2c_2 - c_1$.

b) Assumptions 1-2' and $\sqrt{k} \in [\frac{\alpha}{4}, \frac{\alpha}{2}]$ *where* $\alpha = a - c$.

c) Assumptions 1' and 2" with $\alpha_1 b_2 > \alpha_2 d_1$ *and* $\alpha_1 d_2 < b_1 \alpha_2$ *or with* $2\alpha_1 b_2 < d_1 \alpha_2$ *and* $2\alpha_2 b_1 < d_2 \alpha_1$.

d) Assumptions 1" and 2".

The intersection of BF and Nash equilibrium actions is non-empty.

Proof. Part a) is proved in Lemma 4. Part b) is proved in Lemma 5. Part c) is proved in Lemma 6. Part d) is proved in Lemma 7. All these Lemmas are gathered in the Appendix. ■

We are now prepared to prove our results. Let us begin by a very simple observation. Let $(s_1^N, s_2^N, ...s_n^N)$ be a list of actions that are a Nash equilibrium of the static game (recall that an action can be a quantity or a price). Then we have the following:

Proposition 6. *Assume that the actions corresponding to a one shot NE are BF. Then, the allocation corresponding to this Nash equilibrium can be sustained as a SPNE of the dynamic game.*

Proof. An observation in Fudenberg and Tirole (1991), p. 149, states that if $(s_1^N, s_2^N, ..., s_n^N)$ is a Nash equilibrium of the static game then the open-loop strategies $\sigma_i^* = (s_i^N, s_i^N, ...s_i^N, ...)$ $i = 1, 2, ..., n$ are a subgame perfect Nash equilibrium of the repeated game when there are no bankruptcy considerations. Since with these actions no player is bankrupt, these strategies conform indeed a SPNE of the dynamic game. ■

Let π_i^N be the profits obtained by i in a Nash equilibrium of the one-shot game.

Proposition 7. *Assume that the actions corresponding to a one shot NE are BF. When $\delta \to 1$, any sequence of action profiles which are BF and yield profits larger than π_i^N can be sustained as a Subgame Perfect Nash Equilibrium.*

Proof. Let $(\tilde{s}^1, \tilde{s}^2, ..., \tilde{s}^t, ...)$ be the sequence of action profiles with the desired properties. Consider the following strategy for a generic player, say i: At time 1 play the action $\tilde{s}_i^1$. At time $\tau = 2, 3, ..., t, ...$if the history only includes actions profiles $(\tilde{s}^1, \tilde{s}^2, ..., \tilde{s}^{\tau-1})$ play $\tilde{s}_i^{\tau}$. In any other case play s_i^N. It is clear that such strategies yield the desired sequence of actions. Also, by the usual reasoning (see e.g. Fudenberg and Tirole (1991)) such strategies are a Subgame Perfect Nash Equilibrium when δ is sufficiently close to one. ■

A similar result to Proposition 7 can be obtained in the case of two quantity setting firms with non-decreasing non constant average cost and concave profit functions defined on a compact set.

Let $(\hat{x}_1, \hat{x}_2)$ be such that $\hat{x}_1 = \hat{x}_2 \neq 0$, and $\pi_i(\hat{x}_1, \hat{x}_2) = 0$. In this case the set of BF action profiles is

$$BF = \{(x_1, x_2)/x_i \leq \hat{x}_i \text{ for all } i \in \{1, 2\} \text{ and } \pi_i(x_1, x_2) \geq 0\}. \tag{6.1}$$

(see Corollary 1). Let $BRG_i = \{(x_i, x_j)/x_i \in \arg\max \pi_i(x_i, x_j)\}$. We need the following auxiliary result.

Lemma 3. *For all $i \in \{1, 2\}$, $BRG_i \cap BF \neq \emptyset$.*

Proof. Let $x_i \in \arg\max \pi_i(x_i, \hat{x}_j)$. Trivially, $\pi_i(x_i, \hat{x}_j) \geq 0$. Since $\pi_i()$ is concave, $\hat{x}_1 = \hat{x}_2 \neq 0$, and $\pi_i(\hat{x}_1, \hat{x}_2) = 0$, then $x_i \leq \hat{x}_i$. Furthermore, since $\pi_j()$ is decreasing in x_i, $\pi_j(\hat{x}_i, \hat{x}_j) = 0$, and $x_i \leq \hat{x}_i$, $\pi_j(x_i, \hat{x}_j) \geq 0$. Thus, $(x_i, \hat{x}_j) \in BRG_i \cap BF$. ■

Let $(x_i^i, x_j^i) \in BRG_i \cap BF$ be the solution to $\min_{x_j}[\max_{x_i} \pi_i(x_i, x_j)]$ such that $(x_i, x_j) \in BRG_i \cap BF$. Let $\pi_i^m = \pi_i(x_i^i, x_j^i)$. We call x_j^i the minimax BF action for player j against player i. By Weierstrass theorem, $\max_{x_i} \pi_i(x_i, x_j)$ exists and from Berge theorem (Berge, 1963), this function is continuous in x_j. Thus, $\min_{x_j}[\max_{x_i} \pi_i(x_i, x_j)]$ exists. Now we have the following:

Proposition 8. *Let $(\bar{x}_i, \bar{x}_j)$ be a BF quantity profile such that $\pi_i(\bar{x}_i, \bar{x}_j) > \pi_i^m$. Then there is $\bar{\delta} < 1$ such that for all $\delta \in (\bar{\delta}, 1)$, $(\bar{x}_i, \bar{x}_j)$ can be sustained as a Nash equilibrium of the dynamic game.*

Proof. For convenience let us normalize payoffs as $(1-\delta)\sum_{t=0}^{\infty}\delta^t\pi_i(s^t)$. Consider the following strategy for each player i : Play $\bar{x}_i$ in period 0, and continue to play $\bar{x}_i$ whenever the realized quantity profile in the previous period was $(\bar{x}_i, \bar{x}_j)$. If in a previous period agent j deviates, then play the minimax bankruptcy free action against player j, (x_i^j), for the rest of the game". Observe first that since $(\bar{x}_i, \bar{x}_j)$ is bankruptcy free, no deviation of player j can make agent i bankrupt without bankrupt agent j. Secondly, playing the minimax bankruptcy free action against player j prevents player j to bankrupt player i without ruin himself.

Let us see that no player can gain by deviating from this strategy. If player i deviates, he can only drive out the market player j by ruining himself because $(\bar{x}_i, \bar{x}_j)$ is a bankruptcy action profile. Thus, to ruin player j is not a profitable deviation. So if player i deviates, he takes an action that keep both agents in the market. In the period he deviates, he receives at most $\max_{(x_i,x_j)}\pi_i(x_i,x_j)$. From there on, player j will play the minimax bankruptcy action against player i, x_j^i, and player i will obtain at most π_i^m in all periods after the deviation. Thus, if player i deviation occurs in period t, he will get at most

$$(1-\delta^t)\pi_i(\bar{x}_i,\bar{x}_j) + \delta^t(1-\delta)\max_{(x_i,x_j)}\pi_i(x_i,x_j) + \delta^{t+1}\pi_i^m. \tag{6.2}$$

Let $\bar{\delta}_i$ be such that

$$(1-\bar{\delta}_i)\max_{(x_i,x_j)}\pi_i(x_i,x_j) + \bar{\delta}_i\pi_i^m = \pi_i(\bar{x}_i,\bar{x}_j). \tag{6.3}$$

Since $\pi_i(\bar{x}_i,\bar{x}_j) \geq \pi_i^m$, $\bar{\delta}_i < 1$. Let $\bar{\delta} = \max_i \bar{\delta}_i$, then clearly for all $\delta \in (\bar{\delta}, 1)$, no agent i can deviate profitably. ■

With respect to Proposition 6, Proposition 8 only holds for δ sufficiently close to 1 and two quantity-setter firms. Also, it refers to Nash equilibrium, a weaker notion than SPNE. However, the assumption that the static Nash equilibrium is included in the set of BF allocations is dispensed with.

Now we will show that not all BF allocations can be sustained as a Nash equilibrium of the dynamic game.

Assumption 3. There are two firms with identical decreasing average cost. There is x such that $p(x) = 0$.

Under A3, consider the BF allocation in which both firms produce a positive output (x_1^*, x_2^*) such that $p(x_1^* + x_2^*) = AVC(x_i)$, $i = 1, 2$. Recall that by Proposition 2, $|(AVC(\hat{x}_i))'| < \left|\frac{\partial p(\hat{x}_1 + .. + \hat{x}_n)}{\partial x_i}\right|$, thus $\partial\pi_i(x_1^*, x_2^*)/\partial x_i < 0$.

Proposition 9. *Suppose that Assumption 3 holds. The bankruptcy free allocation (x_1^*, x_2^*) cannot be sustained as a Nash equilibrium of the dynamic game.*

Proof. Suppose that there are some strategies sustaining (x_1^*, x_2^*) as a Nash equilibrium of the dynamic game. Consider that at time t firm i reduces an ε his output. With ε sufficiently small, given that $\partial \pi_i(x_1^*, x_2^*)/\partial x_i < 0$, profits for firm i will be positive in this period. In any subsequent period this firm chooses zero output. With this strategy firm i makes positive profits which contradicts that we are in a Nash equilibrium. ■

We now move to the converse of Propositions 6 and 8. Then, denoting monopoly profits for firm i as π_i^M, we have the following:

Proposition 10. *Let $n = 2$. Suppose that $(s_1^1, s_2^1, ..., s_1^t, s_2^t, ...)$ is a sequence of actions yielded by a Nash Equilibrium such that there is an $\epsilon > 0$ with $\pi_i(s^t) + \epsilon \leq \pi_i^M$ for all $t = 1, 2, ...,$, $i = 1, 2$. Then, when $\delta \to 1$ (s_1^t, s_2^t) is BF for all t.*

Proof. Suppose that in period t, firm 1 chooses an action, say $\tilde{s}_1$, which is not BF. Consider the following strategy for firm 2. In period t chooses an action $\tilde{s}_2$, that drives firm 2 into bankruptcy and chooses the output corresponding to monopoly thereafter. In this case, profits for firm 2 are

$$\pi_2(\tilde{s}_1, \tilde{s}_2) + \delta \pi_2^M + \delta^2 \pi_2^M + ... \tag{6.4}$$

Total payoffs for the sequence $(s_1^1, s_2^1, ..., s_1^t, s_2^t, ...)$ are

$$\pi_2(s^t) + \delta \pi_2(s^{t+1}) + \delta^2 \pi_2(s^{t+2}) + ... \tag{6.5}$$

By the definition of a Nash Equilibrium

$$\pi_2(s^t) + \delta \pi_2(s^{t+1}) + \delta^2 \pi_2(s^{t+2}) + ... \geq \pi_2(\tilde{s}_1, \tilde{s}_2) + \delta \pi_2^M + \delta^2 \pi_2^M + ... \tag{6.6}$$

or

$$\pi_2(s^t) - \pi_2(\tilde{s}_1, \tilde{s}_2) \geq \delta(\pi_2^M - \pi_2(s^{t+1})) + \delta^2(\pi_2^M - \pi_2(s^{t+2})) + ... \geq \delta\epsilon + \delta^2\epsilon + ... = \delta\frac{\epsilon}{1-\delta} \tag{6.7}$$

Clearly, when $\delta \to 1$, the above inequality is impossible contradicting that we were in a Nash equilibrium. ■

The difficulty in the case $n > 2$ is that, after, say, firm i is driven bankrupt by an action of firm j, the strategies of the other firms can be anything. Let us introduce a restriction on strategies that will get rid of this difficulty. We would say that a profile of actions $s = (s_1, s_2, ..., s_n)$ *is weakly implementable from period t on*, if there are strategies $(\sigma_1, \sigma_2, ..., \sigma_n)$ such that they yield s as a Nash equilibrium of the subgame starting at t. We now state a new assumption, called R because it says how firms react to the bankruptcy of a firm or a set of firms: They coordinate on the profile of actions that yields the highest aggregate payoff among all actions which are BF if such actions can be weakly implemented.

Assumption R. *Suppose $\pi_j^t < 0$ for a subset of firms, say B, for some t. Let $\hat{s} \in \arg\max \sum_{i \notin B} \pi_i(s)$, $s \in BF$. If there are strategies that weakly implement $\hat{s}$, firms use these strategies.*

Proposition 11. *Suppose that Assumption R holds, the actions corresponding to a one shot NE are BF. Let $(s^1, s^2, ..., s^t, ...)$ be a sequence of action profiles yielded by a Subgame Perfect Nash Equilibrium such that there is a $\epsilon > 0$ with $\pi_i(s^t) + \epsilon \leq \pi_i^M$ for all $t = 1, 2, ...,, i = 1, 2, ...n$. Assume that firms can make side payments. Then, when $\delta \to 1$, s^t is BF for all t.*

Sketch of the Proof. First we have to prove that the profile of actions $\hat{s} \in \arg\max \sum_{i \notin B} \pi_i(s)$, $s \in BF$ exist (for this, we will need some continuity and boundness assumptions).

Second, from Proposition (7), when $\delta \to 1$ such profile of actions is weakly implementable.

Third, consider that in a SPNE strategies do not yield BF actions at, say, period t. This means that there is a set of firms, say B that can be made bankrupted. Assume for a moment that these firms are bankrupted by, say, firm i. In the continuation, by Assumption R, remaining firms implement the strategies that yield $\hat{s}$. Because firms can make side payments, any remaining firm is now better off (for this we need to show that the condition $s \in BF$ in the aggregate payoff maximization is not binding or, in other words that the actions that maximize aggregate profits are BF). So this strategy yields higher profits for firm i when δ is close to one. Therefore SPNE must yield BF actions every period.

7. APPENDIX

Lemma 4. *Under Assumptions 1-2, a Cournot equilibrium is BF iff* $a \leq 2c_2 - c_1$.

Proof. From a previous result we know that a strategy profile is BF if and only if $\hat{x}_1 > 0$, $\hat{x}_j = 0$ for all $j \in \{2, .., n\}$ and $\pi_1(\hat{x}_1, ..., \hat{x}_n) \geq 0$. Let $a_i \equiv a - c_i$ with $a_i > 0$ for all $i = 1, 2, ..., n$. Let us compute the Cournot equilibrium. The best reply function of firm i is

$$x_i = \max\{0, \frac{a_i - b\sum_{j \neq i} x_j}{2b}\}. \tag{7.1}$$

Let x_i^* be the Cournot equilibrium output of firm i. Clearly $x_1^* > x_2^* \geq .... \geq x_n^*$. It follows from the characterization of BF allocations that the necessary and sufficient condition for a Cournot equilibrium to be BF is $x_2^* = 0$ since in this case the output of firms $3, ..., n$ will be zero as well. The Cournot equilibrium fulfils this condition when

$$\frac{a_2 - bx_1^*}{2b} \leq 0 \Leftrightarrow a \leq 2c_2 - c_1. \tag{7.2}$$

■

Notice that this lemma can be rewritten as follows: A Cournot equilibrium is BF iff there is only one active firm. The latter is a very astringent condition. Now we turn our attention to the case of decreasing average costs.

Lemma 5. *Under Assumptions 1-2'.*
a) If $\sqrt{k} \in [0, \frac{\alpha}{4})$ *no Cournot equilibrium if BF.*
b) If $\sqrt{k} \in [\frac{\alpha}{4}, \frac{\alpha}{2}]$, *all asymmetric Cournot equilibria are BF.*

Proof. Without lost of generality, take $b = 1$. The area where both firms have positive profits is

$$x_2 \leq \alpha - x_1 - \frac{k}{x_1}, \text{ and } x_1 \leq \alpha - x_2 - \frac{k}{x_2}. \tag{7.3}$$

As proved in Proposition 2, the set of BF allocation consists of the following:

$$\{(\frac{\alpha + \sqrt{\alpha^2 - 8k}}{4}, \frac{\alpha + \sqrt{\alpha^2 - 8k}}{4})\} \cup \tag{7.4}$$

$$\cup\{(x_1, 0)/x_1 \in [\frac{\alpha + \sqrt{\alpha^2 - 8k}}{4}, \frac{\alpha + \sqrt{\alpha^2 - 4k}}{2}]\} \cup \tag{7.5}$$

$$\cup\{(0, x_2)/x_2 \in [\frac{\alpha + \sqrt{\alpha^2 - 8k}}{4}, \frac{\alpha + \sqrt{\alpha^2 - 4k}}{2}]). \tag{7.6}$$

Let us compute the best reply of a firm, say 1.

$$x_1 = \frac{\alpha - x_2}{2} \text{ if } x_2 \leq \alpha - 2\sqrt{k}, \text{ and } x_1 = 0 \text{ if } x_2 \geq \alpha - 2\sqrt{k}. \tag{7.7}$$

The symmetric Cournot equilibrium is

$$x_1^* = x_2^* = \frac{\alpha}{3}. \tag{7.8}$$

This equilibrium exists as long as profits are positive, namely $\alpha \geq 3\sqrt{k}$. The asymmetric Cournot equilibria are:

$$(\frac{\alpha}{2}, 0) \text{ and } (0, \frac{\alpha}{2}). \tag{7.9}$$

Using (7.7) it is easy to see that asymmetric Cournot equilibria exists as long as $\alpha \leq 4\sqrt{k}$. From all these calculations we obtain that: a) If $\sqrt{k} \in [0, \frac{\alpha}{4})$ the only Cournot equilibrium is symmetric. It is clear that this equilibrium does not belong to the set of BF allocations in (7.4), (7.5) and (7.6). b) If $\sqrt{k} \in [\frac{\alpha}{4}, \frac{\alpha}{2}]$, there are asymmetric Cournot allocations given by (7.9). Clearly any of these points belong to the set of BF. Indeed suppose not so

$$\frac{\alpha + \sqrt{\alpha^2 - 8k}}{4} < \frac{\alpha}{2}, \tag{7.10}$$

but in this case we will have that $\sqrt{\alpha^2 - 8k} < \alpha$ which is impossible. ■

Notice that the set of BF allocations includes allocations like the famous "Limit Pricing" (Sylos-Labini, 1962) where the output of a firm precludes the entry of other firms.

Finally we consider the case of product differentiation.

Lemma 6. *Under Assumptions 1' and 2".*

a) If $\alpha_1 b_2 > \alpha_2 d_1$ and $\alpha_1 d_2 < b_1 \alpha_2$ the unique Cournot equilibrium is BF.

b) If $2\alpha_1 b_2 > d_1 \alpha_2$ and $2\alpha_2 b_1 > d_2 \alpha_1$ but $\alpha_1 b_2 < \alpha_2 d_1$ and $\alpha_1 d_2 > b_1 \alpha_2$ the unique Cournot equilibrium is not BF.

c) If $2\alpha_1 b_2 < d_1 \alpha_2$ and $2\alpha_2 b_1 < d_2 \alpha_1$ the interior Cournot equilibrium is not BF but the two asymmetric Cournot equilibria are BF.

Proof. We first characterize the set of BF allocations. The boundary of the BF set is a straight line in the space (x_1, x_2). We will consider three cases:

(i) The two straight lines do not cross. This case arises when

$$\alpha_1 b_2 < \alpha_2 d_1 \text{ and } \alpha_1 d_2 < b_1 \alpha_2 \text{ or } \alpha_1 b_2 > \alpha_2 d_1 \text{ and } \alpha_1 d_2 > \alpha_2 b_1. \tag{7.11}$$

In the sequel we will disregard this case.

(ii) The two straight lines cross and

$$\alpha_1 b_2 < \alpha_2 d_1 \text{ and } \alpha_1 d_2 > b_1 \alpha_2. \tag{7.12}$$

In this case BF consists of the point where the two firms have zero profits (call it (x_1^0, x_2^0)) plus all the allocations in which a firm produces more than x_i^0, and the other firm produces zero. Formally, the set BF is:

$$(\frac{\alpha_1 b_2 - d_1\alpha_2}{b_1 b_2 - d_1 d_2}, \frac{\alpha_2 b_1 - d_2 \alpha_1}{b_1 b_2 - d_1 d_2}) \cup ([\frac{\alpha_1 b_2 - d_1 \alpha_2}{b_1 b_2 - d_1 d_2}, \frac{\alpha_1}{b_1}], 0) \cup (0, [\frac{\alpha_2 b_1 - d_2 \alpha_1}{b_1 b_2 - d_1 d_2}, \frac{\alpha_2}{b_2}]). \tag{7.13}$$

In the sequel we will call this case OBF ("odd" BF, "odd" because if α_1 and α_2 are not very different and $b_i > d_j$ this case is impossible).

(iii) The two straight lines cross and

$$\alpha_1 b_2 > \alpha_2 d_1 \text{ and } \alpha_1 d_2 < b_1 \alpha_2. \tag{7.14}$$

In this case BF consists in all outputs such that both firms produce less than x_i^0. Formally, the set BF is:

$$[0, \frac{\alpha_1 b_2 - d_1 \alpha_2}{b_1 b_2 - d_1 d_2}] \times [0, \frac{\alpha_2 b_1 - d_2 \alpha_1}{b_1 b_2 - d_1 d_2}] \tag{7.15}$$

In the sequel we will call this case SBF ("standard" BF because it arises when α_1 and α_2 are not very different from each other and $b_i > d_i$).

We now characterize Cournot equilibrium. The Best Reply functions are:

$$x_1 = \max(0, \frac{\alpha_1 - d_1 x_2}{2b_1}) \text{ and } x_2 = \max(0, \frac{\alpha_2 - d_2 x_1}{2b_2}). \tag{7.16}$$

As before we have two cases:

1: The case where

$$2\alpha_1 b_2 > d_1 \alpha_2 \text{ and } 2\alpha_2 b_1 > d_2 \alpha_1. \tag{7.17}$$

We will call this case SC ("Standard" Cournot, because it arises when α_1 and α_2 are not very different from each other and $b_i > d_j$. It can be shown that the Cournot equilibrium is stable under the best reply dynamics). In this case, the equilibrium is unique and interior and it is given by:[7]

$$x_1 = \frac{2\alpha_1 b_2 - d_1 \alpha_2}{4 b_1 b_2 - d_1 d_2} \text{ and } x_2 = \frac{2\alpha_2 b_1 - d_2 \alpha_1}{4 b_1 b_2 - d_1 d_2}. \tag{7.18}$$

[7] If, say $x_2 = 0$, the best reply of firm 1 is to set $x_1 = \frac{\alpha_1}{2b_1}$. Plugging this number in the best reply of firm 2 we see that firm 2 would like to produce a positive quantity, which is a contradiction.

2: The case where

$$2\alpha_1 b_2 < d_1\alpha_2 \text{ and } 2\alpha_2 b_1 < d_2\alpha_1. \tag{7.19}$$

We will call this case OC ("Odd" Cournot. In this case Cournot equilibrium is unstable under the best reply dynamics). In this case there is an interior equilibrium as given by (7.18) but also two asymmetric equilibria given by

$$(\frac{\alpha_1}{2b_1}, 0) \text{ and } (0, \frac{\alpha_2}{2b_2}).^8 \tag{7.20}$$

We are now ready to prove the Proposition.

a) Since (7.14) holds, (7.17) holds too, so in this case there is a unique and interior Cournot equilibrium. It is clear that the outputs defined in (7.18) belong to the BF set, as defined in (7.15): Indeed suppose that

$$\frac{2\alpha_1 b_2 - d_1\alpha_2}{4b_1b_2 - d_1d_2} \geq \frac{\alpha_1 b_2 - d_1\alpha_2}{b_1b_2 - d_1d_2}, \tag{7.21}$$

this would imply that $2b_1(d_1\alpha_2 - \alpha_1 b_2) + d_1(\alpha_2 b_1 - \alpha_1 d_2) \geq 0$, which is impossible. The same argument holds for firm 2.

b) In this case there is a unique and interior Cournot equilibrium given by (7.18). It is straightforward to check that this pair of outputs do not belong to the set BF as defined in (7.13).

c) It is clear that the interior Cournot equilibrium given by (7.18) does not belong to the set defined in (7.13). Now let us check that the asymmetric equilibria belong to the set (7.13). Indeed suppose that

We assume that $a_i > c_i$, and $b_i, d_i > 0$ for all $i = 1, 2$. Notice that we do not assume $b_i > d_i$. Again, let $\alpha_i = a_i - c_i$. Notice that in this framework, profits are non negative for firm i if and only if $x_i = 0$, or $\alpha_i - b_i x_i - d_i x_j \geq 0$. We now we have the following result.

$$\frac{\alpha_1}{2b_1} \geq \frac{\alpha_1 b_2 - d_1\alpha_2}{b_1b_2 - d_1d_2}, \tag{7.22}$$

this would imply that $d_1(2\alpha_2 b_1 - \alpha_1 d_2) \geq \alpha_1 b_1 b_2$ which is impossible. The same argument holds for firm 2. ■

Lemma 7. *When both firms are identical, demand functions are symmetric and outputs are positive at Bertrand equilibrium, Bertrand equilibrium is BF.*

[8] It is easy to see that under our assumptions here, if, say, firm 1, produces $\frac{\alpha_1}{2b_1}$ the best reply of firm 2 is to produce zero output.

Proof. Let $a_1 = a_2 = a$, $b_1 = b_2 = b$, $d_1 = d_2 = d$, and $c_1 = c_2 = c$. It is easily calculated that in a Bertrand equilibrium,

$$p_1^* = p_2^* = \frac{a + bc}{2b - d}. \tag{7.23}$$

The dotted lines in Figure 6 are

$$\bar{p}_1 = \bar{p}_2 = \frac{a + dc}{b}. \tag{7.24}$$

Suppose that $p_1^* > \bar{p}_1$. Then, from the previous equations, we get that $c(b-d) > a$. But it is easily calculated that $x_i^* = a - (b-d)p_i^* = a - \frac{(b-d)}{2b-d}(a + bc) = \frac{b(a-dc+dc)}{2b-d} > 0 \leftrightarrow a > c(b-d)$. ■

8. References

References

[1] Berge, C. (1963). *Espaces topologiques.* Paris, Dunod. Translated by Patterson, E.M.. *Topological Spaces.* Edinburgh, Oliver and Boyd.

[2] Burns, M. R. (1986). "Predatory Pricing and the Acquisition Cost of Competitors". *The Journal of Political Economy*, 94, 2, 266-296.

[3] Cabral, L. and M. Riordan (1994). "The Learning Curve, Market Dominance, and Predatory Pricing". *Econometrica*, 62, 1115-1140.

[4] Cyert, R. and M. de Groot (1970). "Multiperiod decision models with alternating choice as the solution to the duopoly problem". *Quarterly Journal of Economics* 84, 410-429.

[5] Fudenberg. D. and J. Tirole (1991). *Game Theory,* MIT Press, Cambridge, MA.

[6] Maskin, E. and J. Tirole (1987). "A Theory of Dynamic Oligopoly, III. Cournot Competition". *European Economic Review,* 31, 947-968.

[7] Maskin, E. and J. Tirole (1988a). "A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs". *Econometrica*, 56, 3, 549-569.

[8] Maskin, E. and J. Tirole (1988b). "A Theory of Dynamic Oligopoly, II: Price Competition, Kinked Demand Curves, and Edgeworth Cycles". *Econometrica*, 56, 3, 571-599.

[9] Masso, J. and A. Neme (1996). "Equilibrium Payoffs of Dynamic Games". *International Journal of Game Theory*, 25, 4, 437-53.

[10] Milgrom, P. and J. Roberts (1982). "Predation, Reputation, and Entry Deterrence". *Journal of Economic Theory*, 27, 280-312.

[11] Neyman, A. and S. Sorin (2003, editors). *Stochastic Games and Applications.* Kluwer Academic Press.

[12] Rosenthal, R. and A. Rubinstein (1984). "Repeated Two-Players Games with Ruin". *International Journal of Game Theory*, 13, 3, 155-177.

[13] Roth, D. (1996). "Rationalizable Predatory Pricing". *Journal of Economic Theory,* 68, 380-396.

[14] Shapley, L.S. (1953). "Stochastic Games". *Proc. Nat. Acad. Science*, 39:1095-1100.

[15] Sharfstein, D. and P. Bolton (1990). "A Theory of Predation Based on Agency Problems in Financial Contracting". *American Economic Review*, 80, 93-106.

Benoit, J.P. and V. Krishna (1985). " Finitely Repeated Games". *Econometrica*, 53, 4, 905-922.

[16] Sylos-Labini, P. (1962). *Oligopoly and Technical Progress.* Harvard University Press, Cambridge, Mass, and Oxford University Press, London.