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Allocation Rules on Networks

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June 3, 2009

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Keywords: *Networks, Claims Problems, Constrained Proportional Rule, Constrained Equal Awards Rule, Constrained Equal Losses Rule*

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1 Introduction

The world has become a densely connected network especially for markets and natural resources. Given geographical or infrastructure constraints, it is important to understand how scarce resources should be allocated. An example where such network constraints are essential is water resources. As a result of increasing population and developing economies, there is a growing need for water. Supplies are limited and diminishing due to over exploitation, pollution and global warming. The network constraints lead to huge differences in water availability, not only between countries but also between different regions within the same country. Scarcity of water is becoming a global problem. It disrupts businesses across all sectors and geographies and also has economic, political, environmental and social implications. Governments and private sector play important roles in mitigating the problem. The main action should be water governance for efficient and fair allocation to users.

In the last World Economic Forum, it was noted that water stress is global, but its impacts occur regionally and locally: there are few global markets where water stress will not present considerable challenges for businesses and for wider economic growth -as it already has in Australia, the western and south-eastern United States and the Mediterranean region (WEF, 2008). Water scarcity is an intensifying problem especially for the Southern European Countries and it can only be solved through better planning and cooperation (WATEREGIO, 2007).

Water is carried through rivers or pipelines. It is not economical and in most cases infeasible to carry high volumes of water by other means. Hence, we can depict the water distribution infrastructure as a network between sources and cities which are linked by rivers and pipelines. Various networks can be offered. In a stylized model, we assume that sources are only connected to cities and cities are only connected to sources. We could also consider that sources are only connected to cities but cities can be connected among themselves. Different networks represent possible real-life situations.

Networks have been extensively analyzed in economic theory. Myerson's (1977) seminal contribution is to adapt the cooperative game theory structure to accommodate information about the network connecting players. Jackson and Wolinsky (1996) showed that the Myerson value has a direct extension to network games. Jackson (2005) examines the allocation of value among players in a network. The latter two papers are related to network formation, but Corominas-Bosch (2004) analyzes bargaining between buyers and sellers who are connected by an exogenously given network.

As in Corominas-Bosch (2004), we study a bipartite network. Each source has limited supply of water and each city has demand for water from the sources it is connected to. In many instances, total demand exceeds total supply. The problem of water stress is still present when total supply is enough to satisfy the total demand. Due to network constraints, there could be a part of a network where local demand exceeds local supply. Given the network constraints, the demand of cities, and the supply of sources, the question is how to allocate water resources among the cities. An allocation rule assigns to each city a quantity of water satisfying the following feasibility constraints: First, a city can not receive more than its demand. Second, a source can not deliver more than its supply. The objective is to identify allocation rules that are well-behaved from the normative viewpoint. In addition to efficiency, we look for distributional fairness. It is evident that a detailed analysis of efficient and fair allocations of resources given the network constraints is needed.

When individuals have claims on a resource that sum up more than what is available, how should the resource be divided? This problem is called a claims problem, formally introduced by ONeill (1982). An application is to bankruptcy: when a firm goes bankrupt, how its liquidation value should be divided among its creditors. An allocation rule recommends for each claims problem a division between the claimants of the amount available. The objective of this literature is to identify well-behaved allocation rules. Since there could be a part of a network where local demand exceeds local supply, there is a subset of individuals facing a claims problem. Several rules are commonly used in practice or discussed in theoretical work (Thomson, 2003, 2006). These rules are the proportional rule, the constrained equal awards and the constrained equal losses rules. We define these rules to accommodate network constraints and give algorithms how to calculate these allocation rules. After specifying the model of a network problem, we formulate a list of desirable properties of allocation rules. We give axiomatic characterizations of the constrained proportional rule and the constrained equal awards rule.

In Section 2, we formally introduce the model. In Section 3 and 4, we define rules and give algorithms how to calculate such rules, and we define the properties on rules. In Section 6, we give the main results. In Section 7, we give concluding comments.

2 Model

There are n sources with non-negative supplies $S = \{s_1, ..., s_n\}$ and m cities with non-negative demands $C = \{c_1, ..., c_m\}$. By the abuse of notation, each s_i refers to source i and the supply of source i. Similarly, each c_j refers to city j and the demand of city j.

They are embedded in a network that connects sources to cities which can acquire resources only from the sources they are connected to. We represent the network as a graph. A *non-directed bipartite graph* $g = \langle S \cup C, L\rangle$ consists of a set of *nodes* formed by sources S, and cities C, and a set of *links* L. A *link* connects a source s_i to a city c_j and is denoted by (i, j). A source s_i is *linked* to a city c_j in g if $(i, j) \in L$. We use $(i, j) \in g$ and $(i, j) \in L$ interchangeably, meaning that s_i is linked to c_j in g. Let $r(g)$ be the number of links in g. Let $\mathcal{G}_{n\times m}$ be the set of all bipartite graphs between two sides of n and m nodes.

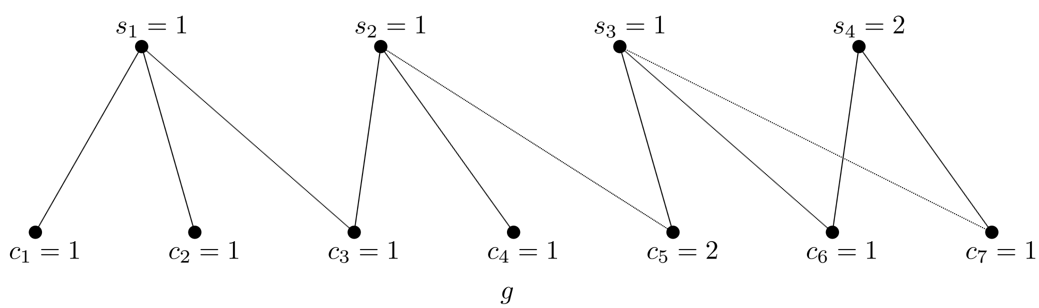

Figure 1

A *subgraph* $g_0 = \langle S_0 \cup C_0, L_0\rangle$ of g is a graph such that $S_0 \subseteq S, C_0 \subseteq C, L_0 \subseteq L$, and each link in L that connects a source in S_0 to a city in C_0 is a member of L_0. Hence, a node of g_0 continues to have the same links it had in g with the other nodes in g_0 and we denote by $g_0 \subseteq g$. For a subgraph g_0 of g, we denote by $g - g_0$, the subgraph of g that results when we remove the set of nodes $S_0 \cup C_0$ from g. Given a subgraph $g_0 = \langle S_0 \cup C_0, L_0\rangle$ of g, let the complete bipartite graph with nodes $S_0 \cup C_0$ be *the completed graph of* g_0 and denote by $\overleftrightarrow{g_0}$. Let $N_g(s_i)$ be the set of cities connected to s_i in $g = \langle S \cup C, L\rangle$, formally, $N_g(s_i) = \{c_j \in C \text{ such that } (i, j) \in g\}$. Similarly, let $N_g(c_j)$ be the set of sources connected to c_j.

An *inclusive subgraph* $g_0 = \langle S_0 \cup C_0, L_0\rangle$ *of* g is such that g_0 is connected and $S_0 = \bigcup\limits_{c_j \in C_0} N_g(c_j)$. An inclusive subgraph includes all the sources to which its cities are connected in graph g. We denote by $W(g) = \{g_0 \subseteq g : g_0 \text{ is inclusive}\}$ the set of inclusive subgraphs in g. Since g is an inclusive subgraph of itself, $W(g) \neq \emptyset$.

An *exclusive subgraph* $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ *of* g is such that for each $s_i \in S_0$, there is no $c_j \notin C_0$ such that $(i, j) \in g$. The cities in an exclusive subgraph do not share their sources with cities outside the subgraph. The sources in g_0 are exclusive to the cities in g_0

In the network g in Figure 2, the subgraph g_1 in the dashed box is inclusive and the subgraph g_2 in the dashed box is exclusive.

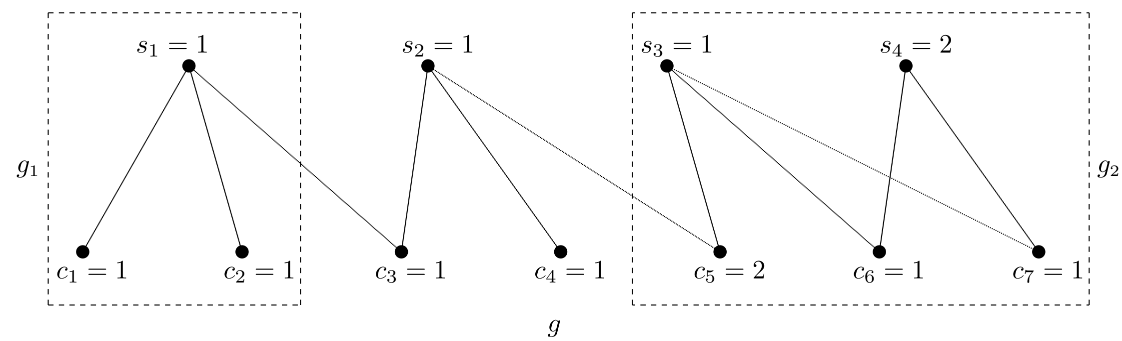

Figure 2

Labeling of pairs (i,j) We first order all possible links such that the links of a city j are assigned a lower number than any source i for $i > j$, and the links of a city are ordered according to the indices of the sources they connect. The label of a possible link (i, j) is denoted by $\tau(i, j)$. For example, for 2 sources and 2 cities, we order the links starting from source s_1 and city c_1, $\tau(1, 1) = 1$. The second link is between s_2 and c_1, $\tau(2, 1) = 2$. Now, as all links of city c_1 are ranked, τ next ranks the link between c_2 and s_1, $\tau(1, 2) = 3$. Then comes the link between city c_2 and source s_2, $\tau(2, 2) = 4$.

For a network g, let $Y(g) = \{y \in \mathbb{N}_+ : y = \tau(i, j) \text{ for some } (i, j) \notin g\}$ be set of indices that τ assigns to links which are not in g. Assume without loss of generality that $|Y(g)| = m \times n - r(g)$, for some $1 \leq r(g) \leq m \times n$, where $r(g)$ is the number of links in graph g.

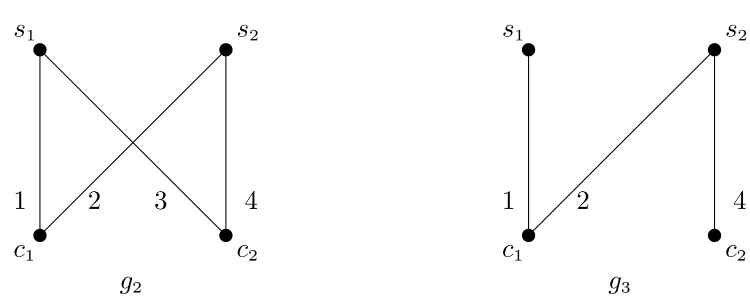

Figure 3

For 2 cities and 2 sources, for a graph g_3 in Figure 2, if the only missing link is $(1,2)$, then $Y(g_3) = \{3\}$ and $r(g_3) = 3$. While $Y(g)$ does depend on g, τ orders all possible links, independent of g. We can see how this works on an example. Suppose that 2 cities and 2 sources, form a completely connected bipartite graph g_2. For graph g_2 in Figure 2, $Y(g_2) = \emptyset$. Now we cut the link between c_2 and s_1, to obtain g_3. Although link $(1,2)$ does not exist in g_3, it is still labeled by τ, i.e., $\tau(1,2) = 3$, meaning that $Y(g_3) = \{3\}$.

Now, we define the column vector that shows the quantities flowing at each link. Let $q_{ij} \geq 0$ be the amount of water extracted by city c_j from source s_i. Let $F = [f_z]$ be the column vector of quantities extracted such that for q_{ij}, the quantity extracted from source s_i by c_j, $f_{\tau(i,j)} = q_{ij}$. For 2 cities and 2 sources:

$$F = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix}$$

Let F_{-j} be the vector obtained by deleting row j from F. For $J \subset \mathbb{N}_+$, let F_{-J} be the vector obtained deleting each row $j \in J$ and column $j \in J$ from F. For $Y(g) \subset \mathbb{N}$, let F_g be the matrix obtained by deleting each row $y \in Y(g)$ from F. F_g is the link by link profile of extractions and has size $r(g)$. For the two graphs given above:

$$F_{g_2} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix} \qquad F_{g_3} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{22} \end{bmatrix}$$

For $j \in \mathbb{N}_+$, let F_{g-j} be the vector obtained from F_g by deleting row j. For $J \subset \mathbb{N}_+$, let F_{g-J} be the vector obtained from F_g by deleting each row $j \in J$. Let $\mathbb{F}_r$ be the set of all non-negative real valued column vectors of size r.

A *resource allocation problem* is a triple $P \equiv (S, C, g) \in \mathcal{P} \equiv \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathcal{G}_{n\times m}$ such that there exists an inclusive subgraph $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ of g for which $\sum_{c_j \in C_0} c_j > \sum_{s_j \in S_0} s_j$.

Example 1. *Let (S, C, g) be a resource allocation problem in Figure 1. $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$, and $(s_1, s_2, s_3, s_4) = (1, 1, 1, 2)$, and $(c_1, c_2, c_3, c_4, c_5, c_6, c_7) =$*

$(1, 1, 1, 1, 2, 1, 1)$.

The set of non-negative flow vectors in (S, C, g) is $\mathbb{F}_g = \mathbb{F}_{r(g)}$. An allocation q is a vector $q = (q_1, q_2, ..., q_m) \in \mathbb{R}^m_+$ that assigns consumption q_j to c_j. We say that a flow vector $F_g \in \mathbb{F}_g$ supports an allocation $q = (q_1, q_2, ..., q_m)$ if for each $j \in \{1, 2, ..., m\}$, $q_j = \sum_{s_i \in N_g(c_j)} q_{ij}$. We say that an allocation q is *feasible* in (S, C, g) if there exists a non-negative flow vector F_g that supports it and for each $s_i \in S$, $\sum_{c_j \in N_g(s_i)} q_{ij} \leq s_i$.

Next, we define an efficient allocation. An allocation q is *efficient* if it is *feasible* and there is no other feasible allocation q' such that for each city c_j, we have $c_j \geq q'_j \geq q_j$ and $\sum_{c_j \in C} q'_j > \sum_{c_j \in C} q_j$. A feasible allocation q satisfies *claim boundedness* if for all c_j, $q_j \leq c_j$.

A *resource allocation rule*, shortly *rule*, φ is a function which assigns to each resource allocation problem (S, C, g) an *efficient* allocation that satisfies *claim boundedness*. Since an Each allocation assigns a consumption to each city c_j, $\varphi_j(S, C, g) = \sum_{s_i \in N_g(c_j)} \varphi_{ij}(S, C, g)$.

An *autarchic subgraph* $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ *of* g is an exclusive subgraph of g such that for each $c_j \in C_0$, $q_j = c_j$ is feasible in (S_0, C_0, g_0). Let $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ be an *autarchic subgraph of* g. Since a rule φ satisfies *efficiency* and *claim boundedness* by definition, for each $c_j \in C_0$, $\varphi_j(S, C, g) = c_j$. In (S, C, g), let $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ be the *maximal autarchic subgraph of* g,[1] then let $(S', C', g') = (S \setminus S_0, C \setminus C_0, g - g_0)$ be the *true resource allocation problem*. In the network g in Figure 4, the subgraph $g - g_0$ is *autarchic subgraph*.

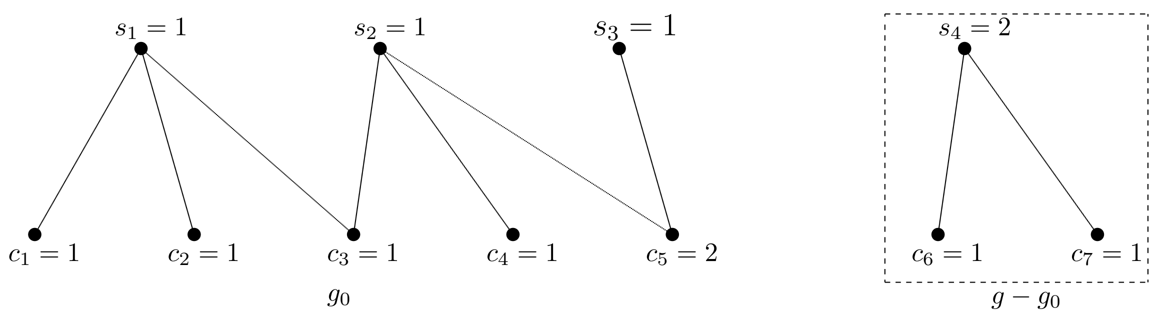

Figure 4

3 Rules

3.1 Constrained Proportional Rule

The first rule we define preserves the proportionality idea given the network constraints.

[1] The *maximal autarchic subgraph* g_0 *of* g is an *autarchic subgraph of* g such that there is no *autarchic subgraph* g'_0 *of* g which contains g_0.

Constrained Proportional Rule, CP : For each $P = (S, C, g) \in \mathcal{P}$, Constrained Proportional Rule chooses an *efficient* allocation $CP(S, C, g) = q$ that satisfies *claim boundedness* such that for each $s_i \in S$, for each pair $\{c_j, c_k\} \subseteq N_g(s_i)$ if $\frac{q_j}{c_j} > \frac{q_k}{c_k}$, then $q_{ij} = 0$.

First, we show that *constrained proportional allocation* $CP(S, C, g) = q$ is unique: Let q and q' be two *efficient* allocations that satisfy *claim boundedness* such that for each $s_i \in S$, for each pair $\{c_j, c_k\} \subseteq N_g(s_i)$ if $\frac{q_j}{c_j} > \frac{q_k}{c_k}$, then $q_{ij} = 0$ and also if $\frac{q'_j}{c_j} > \frac{q'_k}{c_k}$, then $q'_{ij} = 0$. Since $q \neq q'$, there is $c_j \in C$ such that $q'_j > q_j$ and there is $c_l \in C$ such that $q'_l > q_l$. If not, for each $c_l \in C$, we have $q'_l \geq q_l$ contradicting that q is an efficient allocation. There is also a path connecting nodes c_j and c_l. If there is $s_i \in S$ such that $\{c_j, c_l\} \subseteq N_g(s_i)$, then one of these cases is true:

Case 1 $\frac{q_j}{c_j} = \frac{q_l}{c_l}$: It is $\frac{q'_j}{c_j} > \frac{q'_l}{c_l}$. Then, it implies $q'_{ij} = 0$ which is a contradiction.

Case 2 $\frac{q_j}{c_j} > \frac{q_k}{c_k}$: By the definition of *Constrained Proportional Rule*, $q_{ij} = 0$. Then, $q_{ij} > 0$ and $q'_{ik} < q_{ik}$ and $\frac{q'_j}{c_j} > \frac{q'_k}{c_k}$. Then, by the definition of *Constrained Proportional Rule*, $q'_{ij} = 0$ which is a contradiction.

Case 3 $\frac{q_j}{c_j} < \frac{q_k}{c_k}$: By the definition of *Constrained Proportional Rule*, $q_{ik} = 0$ contradicting $q'_{ik} \geq 0$.

If c_j and c_l do not have common source then there is $c_k \in C$ on the path connecting c_j and c_l such that there is $s'_i \in S$ $\{c_k, c_l\} \subseteq N_g(s'_i)$ and the one of cases above is true. Therefore, *constrained proportional allocation* $CP(S, C, g) = q$ is unique.

Next, we show the existence of *constrained proportional allocation* through constructing an algorithm. Before the algorithm, we need some definitions. Given an inclusive subgraph g_0, $\frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j}$ is the proportion of demand to supply in g_0. A *least inclusive subgraph in proportions* $\widehat{g} = \left\langle \widehat{S} \cup \widehat{C}, \widehat{L} \right\rangle$ *of* g is such that

$$\frac{\sum_{s_i \in \widehat{S}} s_i}{\sum_{c_j \in \widehat{C}} c_j} < \frac{\sum_{s_i \in S} s_i}{\sum_{c_j \in C} c_j} \text{ and } \left\langle \widehat{S} \cup \widehat{C}, \widehat{L} \right\rangle \in \underset{\langle S_0 \cup C_0, L_0 \rangle \in W(g)}{\text{argmin}} \frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j}$$

Note that a least inclusive subgraph in proportions of g cannot have a least inclusive subgraph in proportions of its own.

Proposition 1. *For each $P = (S, C, g) \in \mathcal{P}$, the least inclusive algorithm in proportions (MIAP) leads to $CP(S, C, g)$.*

Step 1: Take g. Suppose $g = \langle S \cup C, L \rangle$ has no least inclusive subgraph in proportions. Then, for each $c_j \in C$,

$$MIAP_j(S,C,g) \equiv \min\{\frac{c_j}{\sum_{c_k \in C} c_k} \sum_{s_i \in S} s_i, c_j\}$$

So, suppose $g = \langle S \cup C, L \rangle$ has a least inclusive subgraph in proportions $g_0 = \langle S_0 \cup C_0, L_0 \rangle$.

Then, for each $c_j \in C_0$,

$$MIAP_j(S,C,g) \equiv \min\{\frac{c_j}{\sum_{c_k \in C_0} c_k} \sum_{s_i \in S_0} s_i, c_j\}$$

Step 2: Now, for the rest of the cities and sources apply Step 1 to $g - g_0$. Since, there are finite number of sources and cities, this process ends at a finite time assigning to each city a consumption.

Proof. Let $P = (S,C,g) \in \mathcal{P}$. If g has no least inclusive subgraph in proportions, then the suggested allocation is feasible in $\overleftrightarrow{g}$.

Lemma 1. *Let $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ of g be an inclusive subgraph. If g_0 has no least inclusive subgraphs in proportions, then $MIAP(S,C,g)$ is a feasible allocation in g_0.*

The proof of Lemma 1 is parallel to proof of Proposition 3 in İlkılıç (2008). If g has a least inclusive subgraph g_0, then by Lemma 1, the suggested allocation in g_0 is feasible.
It is clear that $MIAP(S,C,g)$ satisfy claim boundedness. Let q' be a feasible allocation such that for each city c_j, we have $q'_j \geq MIAP_j(S,C,g)$ and $\sum_{c_j \in C} q'_j > \sum_{c_j \in C} MIAP_j(S,C,g)$. Let g_0 be a least inclusive subgraph obtained in MIAP. If $\frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j} < 1$, then $\sum_{c_j \in C_0} MIAP_j(S,C,g) = \sum_{s_i \in S_0} s_i$. If $\frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j} \geq 1$, then for each $c_j \in C_0$, $MIAP_j(S,C,g) = c_j$. Hence, q' does not satisfy claim boundedness. Therefore, q' is not efficient.
Let $\{c_j, c_k\} \subseteq N_g(s_i)$ if $\frac{q_j}{c_j} > \frac{q_k}{c_k}$. Then, in the algorithm c_j and c_k belong to two distinct least inclusive subgraphs $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ and $g'_0 = \langle S'_0 \cup C'_0, L'_0 \rangle$ such that $c_j \in C_0$ and $c_k \in C'_0$. Since $\frac{q_j}{c_j} > \frac{q_k}{c_k}$, $\frac{\sum_{s_i \in S} s_i}{\sum_{c_j \in C} c_j} > \frac{\sum_{s_i \in S'} s_i}{\sum_{c_j \in C'} c_j}$. Hence, according to the algorithm $q_{ij} = 0$. □

Example 2. *Let (S,C,g) be a resource allocation problem in Example 1. For each $c_j \in C$, In Step 1, the least inclusive subgraph in proportions is g_0 as in Figure 4. Then, $\lambda_{g_0} = \frac{1}{2}$ and for each $c_j \in C_0$, $MIAP_j(S,C,g) \equiv \min\{\frac{c_j}{\sum_{c_k \in C_0} c_k} \sum_{s_i \in S_0} s_i, c_j\}$. In Step 2, $g - g_0$*

has no least inclusive subgraph in proportions. Then, $\lambda_{g-g_0} = 1$ *and for each* $c_j \in C \backslash C_0$, $MIAP_j(S,C,g) \equiv \min\{\frac{c_j}{\sum_{c_k \in C} c_k} \sum_{s_i \in S} s_i, c_j\}$. *Therefore, the constrained proportional rule gives* $CP(S,C,g) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)$.

3.2 Constrained Equal Awards Rule

The second rule we define preserves the egalitarian idea given the network constraints.

Constrained Equal Awards Rule, CEA : For each $P = (S,C,g) \in \mathcal{P}$, Constrained Equal Awards Rule chooses an *efficient* allocation $CEA(S,C,g) = q$ that satisfies *claim boundedness* such that for each $s_i \in S$, for each pair $\{c_j, c_k\} \subseteq N_g(s_i)$ if $q_j < c_j$, $q_k < c_k$ and $q_j > q_k$, then $q_{ij} = 0$.

Next, we show the existence of *constrained equal awards allocation* through constructing an algorithm. Before the algorithm, we need some definitions. Given an inclusive subgraph g_0, $\mu_{g_0} \equiv \max_{C' \subset C_0}\{\frac{\sum_{s_i \in S_0} s_i - \sum_{c_j \in C'} c_j}{|C_0| - |C'|}\}$ is the average award in g_0. A *least inclusive subgraph in awards* $\widehat{g} = \left\langle \widehat{S} \cup \widehat{C}, \widehat{L} \right\rangle$ *of* g is such that

$$\mu_{\widehat{g}_0} < \mu_g \text{ and } \left\langle \widehat{S} \cup \widehat{C}, \widehat{L} \right\rangle \in \underset{g_0 \in W(g)}{\operatorname{argmin}} \mu_{g_0}$$

Proposition 2. *For each* $P = (S,C,g) \in \mathcal{P}$, *the least inclusive algorithm in awards (MIAA) leads to* $CEA(S,C,g)$.

Step 1: Take g. *Suppose* $g = \langle S \cup C, L \rangle$ *has no least inclusive subgraph in awards.*

Then, for each $c_j \in C$,

$$MIAA_j(S,C,g) \equiv \min\{\mu_g, c_j\}$$

So, suppose $g = \langle S \cup C, L \rangle$ *has a least inclusive subgraph in awards* $g_0 = \langle S_0 \cup C_0, L_0 \rangle$. *Then for each* $c_j \in C_0$,

$$MIAA_j(S,C,g) \equiv \min\{\mu_{g_0}, c_j\}$$

Step 2: Now, for the rest of the cities and sources apply Step 1 to $g - g_0$. Since, there are finite number of sources and cities, this process ends at a finite time assigning to each city a consumption.

The proof of Proposition 2 is similar to Proposition 1.

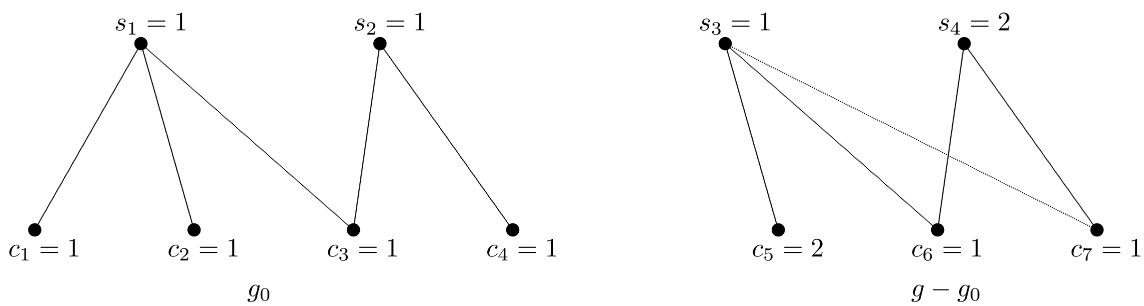

Figure 5

Example 3. *Let (S, C, g) be a resource allocation problem in Example 1. In Step 1, the least inclusive subgraph in awards is g_0 as in Figure 5. Then, $\mu_{g_0} = \frac{1}{2}$ and for each $c_j \in C_0$, $MIAA_j(S, C, g) \equiv \min\{\mu_{g_0}, c_j\}$. In Step 2, $g - g_0$ has no least inclusive subgraph in awards. Then, $\mu_{g-g_0} = 1$ and for each $c_j \in C \backslash C_0$, $MIAA_j(S, C, g) \equiv \min\{\mu_{g-g_0}, c_j\}$. Therefore, the constrained equal awards rule gives $CEA(S, C, g) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)$.*

3.3 Constrained Equal Losses Rule

The third rule we define also preserves the egalitarian idea given the network constraints but it involves the losses the cities incur instead of what they receive.

Constrained Equal Losses Rule, CEL : For each $P = (S, C, g) \in \mathcal{P}$, Constrained Equal Losses Rule chooses an *efficient* allocation $CEL(S, C, g) = q$ that satisfies *claim boundedness* such that for each $s_i \in S$, for each pair $\{c_j, c_k\} \subseteq N_g(s_i)$ if $c_j - q_j < c_k - q_k$, then $q_{ij} = 0$.

Next, we show the existence of *constrained equal losses allocation* through constructing an algorithm. Before the algorithm, we need some definitions. Let $c = (c_1, c_2, .., c_m)$. Given an allocation q, the vector of losses is $\nu = q - c$. Given an inclusive subgraph g_0, $\rho_{g_0} \equiv$

$\min_{C' \subset C_0}\{\frac{\sum_{s_i \in S_0} s_i - \sum_{c_j \in C_0} c_j + \sum_{c_j \in C'} c_j}{|C_0| - |C'|}\}$ is the average loss in g_0. A *least inclusive subgraph in losses* $\widehat{g} = \langle \widehat{S} \cup \widehat{C}, \widehat{L} \rangle$ *of* g is such that

$$\rho_{\widehat{g}_0} < \rho_g \text{ and } \langle \widehat{S} \cup \widehat{C}, \widehat{L} \rangle \in \operatorname*{argmin}_{g_0 \in W(g)} \rho_{g_0}$$

Proposition 3. *For each* $P = (S, C, g) \in \mathcal{P}$*, the least inclusive algorithm in losses (MIAL) leads to* $CEL(S, C, g)$*.*

Step 1: Take g*. Suppose* $g = \langle S \cup C, L \rangle$ *has no least inclusive subgraph in losses. Then, if* $\rho_{g_0} \geq 0$*, then for each* $c_j \in C$*,* $MIAL_j(S, C, g) \equiv c_j$*. Otherwise, for each* $c_j \in C$*,*

$$MIAL_j(S, C, g) \equiv \max\{\rho_g + c_j, 0\}$$

So, suppose $g = \langle S \cup C, L \rangle$ *has a least inclusive subgraph in losses* $g_0 = \langle S_0 \cup C_0, L_0 \rangle$*.*

Then, if $\rho_{g_0} \geq 0$*, then for each* $c_j \in C_0$*,* $MIAL_j(S, C, g) \equiv c_j$*. Otherwise, for each* $c_j \in C_0$*,*

$$MIAL_j(S, C, g) \equiv \max\{\rho_{g_0} + c_j, 0\}$$

Step 2: Now, for the rest of the cities and sources apply Step 1 to $g - g_0$*. Since, there are finite number of sources and cities, this process ends at a finite time assigning to each city a consumption.*

The proof of Proposition 3 is similar to Proposition 1.

Example 4. *Let* (S, C, g) *be a resource allocation problem in Example 1. In Step 1, the least inclusive subgraph in losses is* g_0 *as in Figure 4. Then,* $\rho_{g_0} = -\frac{3}{5}$ *and for each* $c_j \in C_0$*,* $MIAL_j(S, C, g) \equiv \max\{\rho_{g_0} + c_j, 0\}$*. In Step 2,* $g - g_0$ *has no least inclusive subgraph in losses. Then,* $\rho_{g - g_0} = 0$ *and for each* $c_j \in C \backslash C_0$*,* $MIAL_j(S, C, g) \equiv \max\{\rho_g + c_j, 0\}$*. Therefore, the constrained equal losses rule gives* $CEL(S, C, g) = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{7}{5}, 1, 1)$*.*

4 Properties of rules

Now we formulate properties of resource allocation rules and examine their implications. The first is the requirement that if a city's claim increases, then it should receive at least as much as it did initially.

Claims monotonicity: For each $P = (S, C, g) \in \mathcal{P}$, each $P' = (S, C', g) \in \mathcal{P}$ such that $C' = \{c_1, c_2, ..., c'_j, ..., c_m\}$ and $c'_j > c_j$, we have $\varphi_j(S, C', g) \geq \varphi_j(S, C, g)$.

Next, we consider the following situation: Suppose that after an allocation is chosen for some problem, the total supply is found to be α times greater than initially thought at each source. There are two ways to handle this situation. One is to cancel the initial allocation and reallocate the total supply. The other is to let cities keep their initial flows, revise their claims down by these flows, and allocate the incremental amount according to the revised claims. *Composition up* says that both ways of proceeding should result the same allocation vector (Young, 1988).

Composition up: For each $P = (S, C, g) \in \mathcal{P}$, each $\alpha > 1$ such that $P' = (\alpha S, C, g)$ is a resource allocation problem, we have
$\varphi(\alpha S, C, g) = \varphi(S, C, g) + \varphi((\alpha - 1)S, (c_i - \varphi_i(S, C, g))_{i \in C}, g)$.

Now we consider a "dual" property: Suppose that after an allocation is chosen for some problem, the total supply is found to be α times less than initially thought at each source. There are two ways to deal with this situation. One is to cancel the initial allocation and reallocate the revised total supply. The other is to take the initial allocation as the new claims in allocating the revised total supply. *Composition down* says that both ways of proceeding should yield the same awards vector (Thomson, 2003).

Composition down: For each $P = (S, C, g) \in \mathcal{P}$, each $\alpha < 1$, we have $\varphi(\alpha S, C, g) = \varphi(\alpha S, \varphi(S, C, g), g)$.

Next requirement on a rule that is that for each problem, if all the higher claims then city c_j's were reduced to c_j and there would be now enough to compensate every city, then the rule should fully compensate the city c_j.

Conditional full compensation: Let $P = (S, C, g) \in \mathcal{P}$ and $c_j \in C$. Then, consider the set of allocations Q_{c_j} such that for each $q \in Q_{c_j}$, $q_j = 0$, for each $c_k < c_j$, we have $q_k \leq c_k$

and for each $c_k \geq c_j$, we have $q_k \leq c_j$. Take a feasible $q^*(c_j) \in Q_{c_j}$ such that there exists no other feasible $q \in Q_{c_j}$ for which $\sum_{c_k \in C} q_k > \sum_{c_k \in C} q^*_k(c_j)$. Consider $\bar{q}$ such that for each $c_k \in C \backslash \{c_j\}$, $\bar{q}_k = q^*_k(c_j)$ and $\bar{q}_j = c_j$. If $\bar{q}$ is feasible, then $\varphi_j(S, C, g) = c_j$.

Before defining next property, we need some definitions. Given $x \in \mathbb{R}^m$, we denote by $\tilde{x}$ the vector obtained from x by rewriting its coordinates in increasing order. Given x and $y \in \mathbb{R}^m$ with $\sum x_i = \sum y_i$, we say that x *Lorenz-dominates* y, if $\tilde{x}_1 \geq \tilde{y}_1$, $\tilde{x}_1 + \tilde{x}_2 \geq \tilde{y}_1 + \tilde{y}_2$, $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \geq \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3$ and so on, with at least one strict inequality. Given q and $c = (c_1, c_2, ..., c_m)$, let $\lambda(q) = (\lambda_1(q), \lambda_2(q), ..., \lambda_m(q)) = (\frac{q_1}{c_1}, \frac{q_2}{c_2}, ..., \frac{q_m}{c_m})$. The final property says that if the total supply is equal to the half-sum of the claims and there is no *autarchic subgraph*, then the rule chooses an allocation q such that there is no *feasible*, *efficient* allocation q' that satisfy *claim boundedness* such that $\lambda(q')$ Lorenz-dominates $\lambda(q)$.

Midpoint property: For each $P = (S, C, g) \in \mathcal{P}$, if $\sum_{s_i \in S} s_i = \frac{1}{2} \sum_{c_j \in C} c_j$ and there is no *autarchic subgraph* g_0 *of* g, then the rule chooses an allocation q such that there is no *feasible*, *efficient* allocation q' that satisfy *claim boundedness* such that $\lambda(q')$ Lorenz-dominates $\lambda(q)$.

5 Results

First, we show a property of *constrained proportional allocation.*

Proposition 4. *For each $P = (S, C, g) \in \mathcal{P}$, $CP(S, C, g)$ chooses q such that there is no* feasible, efficient *allocation q' that satisfy* claim boundedness *such that $\lambda(q')$ Lorenz-dominates $\lambda(q)$.*

Proof. Let q' be a feasible allocation such that $\lambda(q')$ Lorenz-dominates $\lambda(MIAP(S, C, g))$. Let g_0 be a least inclusive subgraph obtained in MIAP. If $\frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j} < 1$, then $\sum_{c_j \in C_0} MIAP_j(S, C, g) = \sum_{s_i \in S_0} s_i$ and for each city $c_j \in C_0$, $\lambda_j(MIAP(S, C, g)) = \lambda_0$ where $\lambda_0 = \frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j}$. Then, q' is not feasible or there exist $c_k, c_l \in C_0$ such that $\lambda_k(q') < \lambda_0 < \lambda_l(q')$, then $\lambda(q')$ does not Lorenz-dominate $\lambda(MIAP(S, C, g))$. If $\frac{\sum_{s_i \in S_0} s_i}{\sum_{c_j \in C_0} c_j} \geq 1$, then for each $c_j \in C_0$, $MIAP_j(S, C, g) = c_j$. Then, for each c_j, $\lambda_{c_j}(q') > 1$. Hence, q' does not satisfy claim boundedness. Then, $MIAP(S, C, g) = CP(S, c, g)$. □

Now, we give the main result[2].

Theorem 1. *The constrained proportional rule is the only rule satisfying* composition up*,* composition down*, and* midpoint property*.*

Proof. It is clear that the constrained proportional rule satisfies *composition up*, *composition down*, and *midpoint property*. Conversely, let φ be a rule satisfying these properties. If g has autarchic subgraphs, we have $\varphi_j(S,C,g) = c_j$. Then, consider true resource allocation problem (S,C,g). If $\sum_{s_i \in S} s_i = \frac{1}{2}\sum_{c_j \in C} c_j$, then since there is no autarchic subgraph of g and by *mid-point property*, $\varphi(S,C,g) = CP(S,C,g)$. Similarly, by *mid-point property*, $\varphi(\frac{1}{2}S, \frac{1}{2}C, g) = CP(\frac{1}{2}S, \frac{1}{2}C, g)$. Then, let $\sum_{s_i \in S} s_i = \frac{3}{4}\sum_{c_j \in C} c_j$, by *composition up*, $\varphi(\frac{3}{2}S, C, g) = \varphi(S,C,g) + \varphi(\frac{1}{2}S, (c_j - \varphi_j(S,C,g)_{c_j \in C}), g) = \varphi(S,C,g) + \varphi(\frac{1}{2}S, (c_j - CP_j(S,C,g)_{c_j \in C}), g) = CP(S,C,g) + CP(\frac{1}{2}S, \frac{1}{2}C, g) = CP(\frac{3}{2}S, C, g)$. Then, let $\sum_{s_i \in S} s_i = \frac{1}{4}\sum_{c_j \in C} c_j$, by *composition down*, $\varphi(\frac{1}{2}S, C, g) = \varphi(\frac{1}{2}S, (\varphi_j(S,C,g))_{c_j \in C}, g) = \varphi(\frac{1}{2}S, (CP_j(S,C,g))_{c_j \in C}, g) = CP(\frac{1}{2}S, C, g)$. Next step, let for each $c'_j \in C$, $c'_j = \frac{1}{2}c_j$ and apply it to (S, C', g).

By iteration, we calculate for each form $q_{\frac{k'}{2^k}c}$ where $k, k' \in \mathbb{N}$ such that $k' \leq 2^k$, we have $\varphi(S,C,g) = q_{\frac{k'}{2^k}c} = CP(S,C,g)$. The proof concludes by appealing to *continuity* implied by *composition up*. □

Next, we show that *constrained equal awards allocation* Lorenz-dominates each *feasible*, *efficient* allocation q' that satisfies *claim boundedness*.

Proposition 5. *For each $P = (S,C,g) \in \mathcal{P}$, $CEA(S,C,g)$ chooses q such that there is no* feasible*,* efficient *allocation q' that satisfy* claim boundedness *such that q' Lorenz-dominates q.*

Corollary 1. *Let $P = (S,C,g) \in \mathcal{P}$. Then, consider the set of allocations Q_{c_j} such that for each $q \in Q_{c_j}$, $q_j = 0$, for each $c_k < c_j$, we have $q_k \leq c_k$ and for each $c_k \geq c_j$, we have $q_k \leq c_j$. Take a feasible $q^*(c_j) \in Q_{c_j}$ such that there exists no other feasible $q \in Q_{c_j}$ for which $\sum_{c_k \in C} q_k > \sum_{c_k \in C} q^*_k(c_j)$. Consider $\bar{q}$ such that for each $c_k \in C\backslash\{c_j\}$, $\bar{q}_k = q^*_k(c_j)$ and $\bar{q}_j = c_j$. If for $c_j \in C$, $CEA_j(S,C,g) = c_j$, then $\bar{q}$ is feasible.*

[2] Thomson(2006) reports that Toyotaka Sakai (personal communication) provided the same characterization of the constrained proportional rule in claims problems.

Yeh (2006) provided the same characterization of the constrained equal awards rule in claims problems. Here, we extend his result to allocation problems where there are network constraints.

Theorem 2. *The constrained equal awards rule is the only rule satisfying* conditional full compensation *and* claims monotonicity*.*

Proof. It is clear that the constrained equal awards rule satisfies *conditional full compensation* and *claims monotonicity*. Conversely, let φ be a rule satisfying these properties. Let $P = (S, C, g) \in \mathcal{P}$, $x \equiv CEA(S, C, g)$ and $y \equiv \varphi(S, C, g)$. Suppose $y \neq x$. Let q be the allocation obtained by MIAA. For each $c_j \in C$, $x_j = q_j$. By efficiency and claim boundedness, $\sum_{c_j \in C} x_j = \sum_{c_j \in C} y_j$. Let $c_k \in C$, $y_k < x_k$. If $x_k = c_k$, then by Corollary 1, φ violates conditional full compensation. So, suppose $x_k < c_k$. Consider $c' = (c_1, c_2, ..., q_k, ..., c_m)$ and the new problem $P' = (S, C', g)$. By conditional full compensation, $\varphi_k(S, C', g) = q_k$. Then, φ does not satisfy claims monotonicity. □

Proposition 6. *For each $P = (S, C, g) \in \mathcal{P}$, $CEL(S, C, g)$ chooses q such that there is no* feasible, efficient *allocation q' that satisfy* claim boundedness *such that ν' Lorenz-dominates ν.*

6 Results

7 Concluding Comments

The natural resources which require an infrastructure or geographical proximity for their use form a network when multiple users, countries or regions share the same source. Each link in such a network carries an amount of the resource from a source to a user. Users sharing the same source create negative externalities on each other. A network approach makes possible to analyze and solve conflicting claims on scarce resources. The distribution of water resources can be seen as a particular case of the distribution of scarce resources.

The main goal of the project is to build a theoretical framework for distribution of water resources given network constraints.

Although water is usually not traded between countries, many rivers traverse multiple countries. We can use these rivers to distribute water efficiently both within and across countries. Next, we can ask whether the construction of new pipelines can improve distributional efficiency. We can see how regions and countries can share water to decrease the risk of scarcity.

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