Unique equilibrium in a dynamic model of speculative attacks

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UNIQUE EQUILIBRIUM
IN A DYNAMIC MODEL OF SPECULATIVE ATTACKS

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1. Introduction

One of the main challenges of modelling speculative attacks on currency pegs is to explain their puzzling timing. The “second generation” models, due to Obstfeld [24, 25], suggest that murky principles may govern the time of collapse of a currency peg. In these models, a self-fulfilling crisis is understood as a sudden, exogenous jump in market sentiment, from a “good” to a “bad” equilibrium—a shift on financial markets from euphoria to panic. While this perspective does justice to the mystifying timing of crises, it is of no help to explain it. Is it possible to arrive at a theory that explains switches between these states over time? In an article that argues for the merits of the second generation approach, Obstfeld and Rogoff hopefully answer that “future research may succeed in pinning down more precisely the timing of speculative attacks without resorting to equilibrium indeterminacies” [26].

In this paper, we take up this issue by embedding a bare-bones second generation currency crisis model in a dynamic context. In the model, time is continuous and agents

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continuously evaluate whether or not they want to speculate against a currency peg, while the fundamentals of the economy evolve. Like in second generation models, for a wide range of fundamentals, strategic interaction between speculators and a policy maker gives rise to two possible states: tranquillity or panic. Small differences in the speed of agents’ responses to the evolving fundamentals induce an unambiguous timing of the transition between the two states. Thus, the model endogenously accounts for sudden switches in the “equilibrium” that the market “selects” over time. We derive the mathematical conditions for which a shift between states takes place. The shift occurs when economic fundamentals deteriorate beyond an endogenously derived threshold. We characterise this threshold for a natural class of stochastic processes driving economic fundamentals, and derive comparative statics.

The model draws on work by Burdzy et al. [6] and Frankel and Pauzner [12]. In Burdzy et al. [6], agents repeatedly play a stage game with strategic complementarities and multiple equilibria against randomly chosen opponents. Over time, the game slightly changes, while agents are locked into their action for a small period of time. Under these assumptions, agents follow a unique rationalisable strategy, and it is possible to predict theoretically which equilibrium of the game will be played at each point in time. Frankel and Pauzner [12] apply these results in the context of business cycles.

Strategic complementarities are also at the heart of currency crises, but a model of a stage game that is played repeatedly makes less sense in this context. During a currency crisis, speculators interact over a longer period of time with a policy maker aiming to defend a currency peg. Complementarities arise because speculators need to attain a critical mass to force the policy maker to devalue. Speculators may be locked into their choices, their short or long positions in a weak currency, for a (brief) period. Because taking a short position is costly, agents need to forecast evolving fundamentals, the future behaviour of the policy maker, and the speculative pressure resulting from the actions of other speculators to get the timing of an attack right. These are the main elements underlying the model in this paper.

Technically, the model in this paper is closely related to a currency crisis model of Guimarães [17] based on similar elements. Guimarães extends the first generation model (Krugman [20] and Flood and Garber [11]) with the assumption that agents may be locked into their positions due to substantial frictions on asset markets. In his model, an evolving fundamental follows a Wiener process with drift, and speculators attack when it crosses a critical threshold. In this paper, we further extend this result, and improve its utility and generality by enhancing the realism of the model and giving a characterisation result for the threshold. Perhaps more importantly, we also make a significant conceptual contribution by showing that the extended model is more closely related to a strand of “global game”, second generation currency crises models than to the first generation model. Global game models have been prominent in recent currency crises literature (see e.g. Refs. [21], [15], [18]).
The technical and conceptual contributions of this paper—and the difference with the literature so far—may be summed up more thoroughly in the following four points. First, we exploit the presence of strategic complementarities to solve our currency crisis model using the method of Burdzy et al., and familiar from the global games literature. This approach gives a tight correspondence between the mathematical solution of the model and the concept of rationalisable behaviour (Bernheim [4] and Pearce [27]). Because of the form of the first generation model, Guimarães [17] cannot place strategic complementarities at the heart of the analysis and therefore he cannot apply the same argument. The method he uses to solve his model does not lead to the conclusion of a unique, rationalisable strategy for speculators.

Second, the Wiener process with drift considered in Ref. [17] can only describe continuous paths for economic fundamentals, and may not be an adequate model for economic fundamentals. In reality, one observes shocks and discontinuities in data about economic fundamentals. Such discontinuities are potentially important drivers of currency risk, since shocks may trigger large and sudden shifts on financial markets. To model the effects of shocks more clearly, we consider the class of “jump diffusion processes” in this paper. These have been used more widely in the literature to take the discontinuities observed in reality into account. For instance, Jorion [19] applies them to analyse the fundamentals driving foreign exchange markets.

Third, we focus on the case where frictions are small, instead of when they are substantial. While it seems natural to assume there is some heterogeneity in agents’ response times to shocks, in reality this heterogeneity may well be small, perhaps in the order of minutes or even seconds. We derive the properties of our model for the limit case of vanishing frictions to show that (at least for the class of diffusion processes considered) our main qualitative results do not depend on the degree of inertia. In particular, we characterise the critical threshold in the limit, and show that it obeys intuitive comparative statics.

Finally, the results in this paper do not just show in detail how to apply the arguments of Burdzy et al. in the currency crisis setting, but also how to generalise many of the insights of static global game currency crisis models (in the spirit of Morris and Shin [21]) to a dynamic context. Global game models perturb the information structure of a static second generation models to achieve unequivocal equilibrium selection. The few existing articles on global games that attempt to apply this approach in a dynamic context (e.g. Angeletos et al. [1]) show that it is not obvious that this technique is helpful there. This is because a dynamic model gives rise to endogenous learning which complicates the subtle role of information in static global game models. The approach based on heterogeneity in response times to shocks instead of perturbing the information structure circumvents these issues.
Although the information structure that is typical for global games is absent in the model in this paper, the new results indicated above reveal a number of striking parallels with static global game models. Firstly, the model here shares their most acclaimed feature, the existence of a unique rationalisable strategy for speculators that reproduces the sudden shifts from tranquillity to a speculative attacks observed in reality. Secondly, following the approach of Burdzy et al., the existence of this strategy is established using an infection argument in an underlying type space. And thirdly, agents’ behaviour takes the form of a threshold strategy, which we can characterise in terms of the strategic uncertainty associated with speculation. All these findings have almost exact counterparts in static global game models of currency crises.

2. The Model

Consider an economy with a currency peg in place. There are an infinite number of identical, risk neutral speculators indexed on \( N = [0, 1] \). Time is continuous and periodically each of these speculators reviews the decision whether or not to speculate against the pegged currency by taking a short position against it. A speculator that participates in a successful speculative attack earns a positive payoff normalised to 1. The cost of holding a short position is equal to the length of time the position is held times a positive interest rate differential, which is \( r \) per unit of time.\(^1\)

There is a policy maker who, at each point in time \( t \in \mathbb{R}_+ \), faces the decision to abandon the currency peg or to keep it. We assume she decides according to an instantaneous net benefit function that describes the trade-off between keeping and abandoning the peg. As in the static global game model of Morris and Shin [21] (and also their dynamic model [22]) the policy maker’s net benefit depends on two factors: \( \lambda_t \), the fraction of speculators attacking the currency, and \( u_t \), a stochastic component that evolves over time. This component describes the economic fundamentals that affect the policy maker’s trade-off between the benefits and costs of the peg (e.g., unemployment, or real overvaluation of the currency). An increase in \( u_t \) means a worsening of fundamentals; an increase of \( \lambda_t \) reflects more speculative pressure, and thus higher costs of maintaining the currency peg. Now, let net benefit at time \( t \) be given by:

\[
\mathcal{B}_t := B(u_t, \lambda_t),
\]

\( t \in \mathbb{R}_+, \quad B_{ut} \leq c < 0, \quad B_{\lambda t} \leq c < 0, \) for some negative constant \( c \).

In this expression \( B \) is a time-invariant instantaneous benefit function that is continuously differentiable. By applying the implicit function theorem to \( B \) we may characterise the policy

\(^1\)This payoff structure resembles the one adopted usually adopted in the global game literature.
maker’s behaviour at each point \( t \) in time in the following way: the policy maker devalues if the stochastic variable \( u_t \) crosses a certain threshold value for \( u_t \) that depends on \( \lambda_t \). This threshold is given by a time-invariant, continuously differentiable, strictly decreasing and invertible function \( u^* : [0, 1] \to \mathbb{R} \) with compact domain \([\ell, h]\). The policy maker’s threshold curve is illustrated in figure 1.

We assume that \( u_t \) evolves stochastically over time. Concretely, we let:

\[
(2) \quad u_t = W_t + J_t,
\]

where \( W_t \) is a the standard Wiener process, and \( J_t \) is a compound Poisson or “jump” process, with jumps arriving at rate \( \vartheta \). The first component of the process, \( W_t \), reflects small changes in \( u_t \) that occur continuously over time. The second component, the Poisson distributed jumps, reflect large shocks that only occasionally hit the fundamental \( u_t \). (A more general interpretation of \( u_t \) would be that the variable represents information about some relevant economic fundamentals. In this case, the jumps can be interpreted as larger news announcements or the occurrence of important events). We will assume that the large shocks are independently and identically distributed and drawn from a continuous probability distribution with support \( \mathbb{R}_+ \). As emphasised in the introduction, this kind of “jump diffusion” process is quite often adopted as a model for fundamentals that underlie asset prices. In fact, the process in equation (2) is well-studied in a literature on risk theory that originates with Dufresne and Gerber [10].

\[\text{Figure 1: The optimal policy and the impact of speculative activity}\]
The stochastic process in equation (2) is more general than those considered by Burdzy et al. [6] and Frankel and Pauzner [12], and more general than the processes adopted by other global game-like papers. Usually, a process of the simpler continuous form $u_t = W_t$ is investigated, notably in Morris and Shin [22] and Guimarães [17].\(^3\) However, because the standard Wiener process has continuous paths, using the simpler form means that when speculators are able to adjust portfolios very quickly they face virtually no risk of $u_t$ crossing $u^*$ before they are able to revise them. It is questionable whether modelling fundamentals in this way adequately reflect the risk and uncertainty associated with currency speculation. This observation motivates the more general approach in equation (2). We come back to the implications in detail in section 5.

Suppose for the moment that all speculators are able to adjust their short positions instantaneously. In this case, the above setup will inherit the possibility of multiple equilibria from traditional second generation currency crisis models. To see this, consider the point indicated by the black point in figure 1, which denotes a designated state of the system, and now suppose that, suddenly, all speculators decide to attack. In this case, the state of the system will jump to the point indicated above the black point (so that $\lambda_t = 1$). This induces the policy maker to abandon the currency peg and thus will lead to a full-fledged currency crisis. Such a crisis may be labelled “self-fulfilling”, since different behaviour of the speculators would have resulted in a different outcome. Specifically, if all speculators decide to refrain from attacking, the state of the system will jump to the point indicated below the black point, so that $\lambda_t = 0$, and this results in a tranquil situation.

Both jumps are ex post justifiable, since speculators have made a choice consistent with the actual outcome that obtains. Hence, the model exhibits an indeterminacy, that extends to any situation when $u_t$ is in the region between $\ell$ and $h$. However, we will show that this ambiguity only arises if we allow for instantaneous adjustment of positions. Obviously, however liquid an asset, completely instantaneous portfolio adjustments of all speculators at the same time are an abstraction from reality, and in this case it is not an innocent abstraction.

3. Inertia

In reality it is not plausible that large groups of agents move at exactly the same point in time. In reality, portfolio adjustments will exhibit some inertia, perhaps induced by small frictions in financial markets, such as monitoring costs or transaction costs, or simply by the time it takes to place or execute an order. In this section, we introduce a mechanism to model this inertia, based on ideas in Frankel and Pauzner [12]. In the next section,

\(^3\)Incorporating a drift term like is done in Ref. [17] would not substantially change the results in this paper.
we show it removes the ambiguity discussed above. Of course, in reality, inertia may well be very small, and on liquid financial markets it may be in the order of minutes or even seconds. This is especially true during currency crises, when all agents follow the market very carefully. The main qualitative results in this article do not depend on the degree of inertia, only on the fact that there is some inertia, however small. To substantiate this claim, section 5 investigates the properties of the model for the limit case of vanishing inertia.

To model a small degree of inertia, we will assume that speculators do not make revision decisions at every moment in time. Instead, each agent makes revisions at idiosyncratic moments in time that we refer to as her “revision opportunities”. At each revision opportunity, a speculator can decide whether she wants to take a short position or not, and she is then “locked” into this choice until she receives her next revision opportunity. For any given speculator, we assume that the revision opportunities arrive following a Poisson process with parameter $\zeta > 0$. That is, the time $\tau$ elapsed between revision decisions of any given agent is independent, identically distributed, and has cumulative density function $1 - \exp(-\zeta \tau)$. If an agent receives a revision opportunity at time $t$, and the next at $t + \tau$, we refer to the time span $[t, t + \tau)$ as her lock-in period. If the agent decides to speculate against the pegged currency at time $t$, the opportunity cost associated with this amounts to $\tau r$, while the revenue from speculation is 0 if the peg survives the time span $[t, t + \tau)$ and 1 if it collapses during this spell. In expectation, the length of the lock-in period is given by $1/\zeta$. The expected opportunity cost associated with taking a short position is then given by $r/\zeta$. Note that the way inertia is modelled reflects how this is done in other areas of economics. Particularly, under Calvo-style price setting, in each period every price setter sets a new price with a certain probability; in the present model, each speculator revises her portfolio with a certain probability. The approach taken shares Calvo’s “agnostic” attitude towards the origin of inertia, which is modelled in the simplest conceivable way as to keep the model tractable.

Inertia introduces a specific component into the model, namely that speculators respond somewhat heterogeneously to changes in $u_t$. This heterogeneity remains even when revision opportunities arrive very fast. Moreover, an inert revision process allows us to capture two key ideas that are important from a conceptual perspective. First, because behaviour is slightly inert, there is the possibility that an individual speculator fails to take a short position before the peg collapses. Speculators have to weigh this risk against incurring the costs of short positions while the currency does not collapse. This feature of the model captures a realistic and important aspect of speculation, which clearly is important even on very liquid financial markets. The second element is that behaviour can be conditioned on the observation of what other speculators are doing. A speculator can take into account the positions taken by others when making her decision to attack or not. Adding the possibility of “keeping an eye on what the market is doing” constitutes a step towards added realism
Figure 2: Infection in \((u, \lambda)\)-space

over static currency crises models. We assume that when an agent receives the opportunity to revise positions, she perfectly observes the current values of both \(\lambda_t\) and \(u_t\).

4. Unique Equilibrium

Apart from the inertia induced by the revision process, the behaviour of agents in the model is fully rational and firmly forward looking. Speculators will try to determine what is the \textit{optimal} decision, taking the possibility of a future collapse of the peg into account and the possible behaviour of other speculators. In fact, the model exhibits a very straightforward and natural kind of strategic interaction, in which the behaviour of agents in “extreme” states of the system infects the behaviour of agents in other states, that is familiar from the static global games literature, and that closely resembles the argument given in Frankel and Pauzner [12] even though the currency crisis model setup is different from the pure coordination setup considered by these authors. This structure allows us to derive the optimal strategies for speculators.

Consider the situation of a speculator who has a revision opportunity at time \(t\) at some point \((u_t, \lambda_t)\) just to the left of the curve \(u^*\) in figure 2. When deciding whether to speculate or not, she has to anticipate the possibility of a future policy discontinuity, namely the collapse of the peg that takes place if \(u\) crosses the curve \(u^*\) at some time \(\hat{t} > t\). The likelihood of this event depends on two things. First, given that \(u_t\) evolves stochastically over time, it depends on the horizontal distance between \(u_t\) and the curve \(u^*\). Second, because of the shape of the policy function \(u^*\), for identical future paths of the variable \(u_t\), higher paths of \(\lambda_t\) hasten a collapse. The dynamics of \(\lambda_t\) are driven by the decisions of speculators over time. Particularly, if in the near future more speculators would commit to speculating against the pegged currency, speculation becomes more attractive at time \(t\) because \(\lambda_t\) increases, and for any given \(u_t\) when \(\lambda_t\) increases the point \((u_t, \lambda_t)\) always moves closer to the curve \(u^*\). In sum, every individual speculator is confronted with two kinds of uncertainty: the first kind is due
to the nature of the random process, and the second results from the strategic uncertainty about the decisions of other speculators.

Now, on the one hand, when \( u_t \) moves closer and closer to \( u^* \), the probability of a quick collapse goes to one, and, if an agent would opt to refrain from speculation when \( u_t \) is close to \( u^* \), it is likely that she will be stuck with this decision when an actual collapse takes place. Observe that this is true even if all other speculators refrain from attacking after time \( t \); indeed, this holds independently of the decision rules followed by other speculators, so independent of the strategic uncertainty.

On the other hand, if the value of \( u_t \) is very far from \( u^* \), a new revision opportunity will arrive almost certainly before \( u_t \) touches \( u^* \). Since there are opportunity costs associated with speculation against the pegged currency, a speculator who receives a revision opportunity very far from \( u^* \) would prefer to refrain from speculation. In sum, we can bound off a region of \((u_t, \lambda_t)\)-space with a curve \( H_0 \), to the right of which we can be sure that, for any given speculator, to decide to speculate is optimal independent of the decision rules of other speculators; and similarly, there exists a region to the right of some curve \( L_0 \) to the left of which it is optimal to refrain from speculation, independent from the decisions of other speculators. These curves are indicated in figure 2. Following the literature on global games (see e.g. Frankel et al. [13]), we call these two regions dominance regions—because in these regions, one of the choices dominates the other (in the game-theoretic sense; see [14, Definition 1.1]).

The dominance regions provide the starting points for an infection argument. Consider the situation of a speculator who has to decide at a point \((u_t, \lambda_t)\), just slightly to the left of the curve \( H_0 \). She deduces that every speculator who will receive a revision opportunity to the right of \( H_0 \) will opt to speculate. Moreover, since \( u_t \) is close to \( H_0 \), she thinks it is very likely that the stochastic process spends some time to the right of this curve during her lock-in period. When given a revision opportunity to the right of the curve \( H_0 \), the speculator would be willing to speculate against the pegged currency regardless of what other agents would do. But now that she can take into account that colleagues receiving revision opportunities while \( u_t \) is to the right of \( H_0 \) will also choose to speculate, she knows that when \( u_t \) is in this region \( \lambda_t \) will increase. Due to the shape of \( u^* \), under the resulting dynamics of \((u_t, \lambda_t)\) the probability of a collapse of the peg will increase. This makes speculation more attractive; indeed for some values of \((u_t, \lambda_t)\) to the left of \( H_0 \), even if they are not in the original dominance region, the knowledge that others will speculate to the right of \( H_0 \) will be enough to tip the balance unequivocally in favour of speculation in this second step of reasoning at \((u_t, \lambda_t)\). Thus, in what may be called a “second level” of reasoning (which conditions on the earlier observation that to the right of \( H_0 \) agents prefer to speculate), we can expand the region in which we can be sure that the safe asset will be chosen a bit to the left, choosing
a new curve $H_1$ to the left of $H_0$ to bound it. A similar argument works with respect to the
curve $L_0$, and here we find a new curve $L_1$ just to the right of $L_0$.

Repeating the argument, in a next step we can find curves $H_2$ and $L_2$, and so on. Eventu-
ally, iterative expansions of the areas both on the left and on the right leads to two curves $\overline{L}$
and $\overline{H}$ that can be thought of as the limits of the families $H_0, H_1, H_2, \ldots$ and $L_0, L_1, L_2, \ldots$
(a formal argument is given in lemma 4 in the appendix). Following the above reasoning,
to the right of $\overline{H}$, agents always prefer to speculate, and to the left of $\overline{L}$, agents prefer to
refrain from speculation. We will go on to prove that the curves $\overline{H}$ and $\overline{L}$ in fact coincide.
This gives a complete characterisation of the optimal strategies of speculators. Formally:

**Theorem 1.** There is a unique rationalisable strategy, described by a (Lipschitz continuous)
function $Z^* : [0, 1] \rightarrow \mathbb{R}$, such that speculators refrain from speculation if $u_t < Z^*(\lambda_t)$ and
attack if $Z^*(\lambda_t) < u_t$.

The surviving strategy is the unique rationalisable strategy since in each of the steps
that we take to derive the curves $\overline{L}$ and $\overline{H}$, we describe the optimal decision at time $t$
for some combinations of $\lambda_t$ and $u_t$—and deleted suboptimal, or “dominated”, actions—
conditional on the event that such combinations are reached at time $t$, but without reference
to any other aspect of the history of the game. Formally, we apply iterative deletion of
conditionally strictly dominated actions (Fudenberg and Tirole [14], definition 4.2), which is
the continuous time counterpart of iteratively deleting dominated strategies. Since a strategy
that specifies (iteratively) dominated actions at some point in time is never rationalisable,
the surviving strategy must be the only rationalisable strategy in the model.

As a consequence of the theorem, the dynamics of $\lambda_t$ are such that speculative activity is
increasing whenever the state of the system $(u_t, \lambda_t)$ is in the region to the right of the graph
of $Z^*$, but decreasing to the left. The dynamics thus bifurcate abruptly around the graph of
\[ \lambda_t = 1 \]

\[ \lambda_t = 0 \]

Figure 4: Unique Equilibrium

\[ Z^* \], which we (therefore) call the equilibrium threshold. The dynamics are indicated by the arrows in figure 3.

As in Burdzy et al. [6], and Frankel and Pauzner [12], theorem 1 may be proved using a “translation” argument. Since this argument is well-known from the literature on global games, we will only sketch it here. Consider the limiting curves \( \mathcal{L} \) and \( \mathcal{H} \). By lemma 4 in the appendix, these curves are well-defined, and \( \mathcal{L} \) and \( \mathcal{H} \) are continuous functions from \([0, 1]\) into \( \mathbb{R} \). The lemma shows that agents are indifferent when choosing on the graph of \( \mathcal{L} \) when they believe that all other speculators attack to the right of this graph and refrain to the left. Similarly, agents are indifferent when choosing on the graph of \( \mathcal{H} \) and they believe that all other speculators attack to the right of this graph and refrain to its left.

Now suppose that \( \mathcal{L} \neq \mathcal{H} \). Then we may assume that \( \mathcal{L} \) lies everywhere to the left of \( \mathcal{H} \). This means that by “shifting” \( \mathcal{H} \) to the left we can construct a new curve \( \mathcal{S} \) that (i) lies everywhere to the left of \( \mathcal{L} \) and (ii) touches \( \mathcal{L} \) in at least one point \((u_t, \lambda_t)\). This kind of leftward shift of \( \mathcal{H} \) is illustrated graphically by the curve \( \mathcal{S} \) in figure 4. Now first, consider an agent that has to decide at the state \((u_t, \lambda_t)\) lying on the curve \( \mathcal{H} \). This agent must be indifferent when others are using the strategy described by \( \mathcal{H} \), so her expected payoff from attacking the pegged currency is 0. We will compare her payoff to an agent deciding at the state \((u_t, \lambda_t)\) on the “shifted” curve \( \mathcal{S} \) when agents are playing according to the strategy described by \( \mathcal{S} \). For this, we will consider the dynamics of the system depicted in figure 4 under identical relative dynamics for the stochastic process \( u_t \), and compare a path starting from \((u_t, \lambda_t)\) with one starting from \((u'_t, \lambda_t)\). Under each of these paths, the induced dynamics for \( \lambda_t \) must be exactly the same, since \( \mathcal{S} \) and \( \mathcal{H} \) have the same shape. If the curves \( \mathcal{L} \) and \( \mathcal{H} \) do not coincide, then \( u_{t'} < u'_{t'} \) for all \( t' > t \), and a collapse of the currency peg on the path starting from \((u_t, \lambda_t)\) implies an even earlier collapse on a path starting from \((u'_t, \lambda_t)\). Indeed, since \( \mathcal{S} \) is strictly to the left of \( \mathcal{H} \), we conclude that expected payoff from attacking at the state \((u'_t, \lambda_t)\) under the strategy \( \mathcal{H} \) must be strictly greater than the expected payoff.
at \((u_t, \lambda_t)\) under the strategy \(\overline{S}\). Denoting the latter by \(\pi(u_t, \lambda_t; \overline{S})\), we therefore have:

\[
\pi(u_t, \lambda_t; \overline{S}) < 0.
\]

Now compare \(\overline{S}\) with \(\overline{L}\). Since \(\overline{S}\) lies to the left of \(\overline{L}\) we see that for any given path of \(u_t\), at each moment \(t' > t\), weakly more agents attack when all of them use the strategy \(\overline{S}\) instead of \(\overline{L}\). This can only hasten a collapse of the currency peg. From this we conclude that the expected payoff \(\pi(u_t, \lambda_t; \overline{S})\) from attacking at the state \((u_t, \lambda_t)\) when agents follow the strategy described by \(\overline{S}\) is higher than expected payoff at the same state \((u_t, \lambda_t)\) when agents use the strategy described by \(\overline{L}\). Denote this latter payoff by \(\pi(u_t, \lambda_t; \overline{L})\). Now, an agent that has to decide at the state \((u_t, \lambda_t)\), lying on the curve \(\overline{L}\), is indifferent under strategy the described by \(\overline{L}\), so that \(\pi(u_t, \lambda_t; \overline{S}) \geq \pi(u_t, \lambda_t; \overline{L}) = 0\). Hence we have:

\[
0 \leq \pi(u_t, \lambda_t; \overline{S}) < 0,
\]

which is absurd. From this, we conclude that \(\overline{L} = \overline{H}\) after all.

5. Limiting Dynamics and Strategic Uncertainty

In the previous section we showed that the model has a unique equilibrium in the presence of frictions, that is, under the condition that \(\frac{1}{\zeta} > 0\). Frictions on liquid financial markets are generally deemed to be of only small importance, suggesting that the most relevant scenario to study is the case when frictions are small. Therefore, in this section, we study the properties of the model’s equilibrium in the limiting case of vanishing frictions. We now index the equilibrium threshold by \(\zeta\) and will show that as \(\zeta \to \infty\) the shape of the equilibrium threshold \(Z^*_\zeta\) converges to a solution that is determined by the risk associated with the speculation combined due to the large shocks.

To derive the limiting properties of the model, we draw on a number of highly non-trivial results in Burdzy et al. [5] on dynamical systems of the form in figure 3. This Ref [5] is a mathematical companion paper to Frankel and Pauzner [12]. As stated before, the main difference between the model considered in this paper and the model considered by Frankel and Pauzner is that here we consider the possibility of large shocks hitting \(u_t\) (due to the jumps of the compound Poisson process). These shocks have quite important implications in the currency crisis context. This is because, when frictions vanish, during an agent’s lock-in period, a crisis is most likely to be triggered by a such shock, causing the fundamental to cross the threshold after disturbing the dynamical system in figure (3). In order to apply the theory in [5] we will have to incorporate the possibility of large shocks in our arguments.

We start by characterising the dynamics of \(\lambda_t\) when agents behave optimally, but away from the limit. Consider the dynamical system depicted in figure 3. Because of the strategies
used by agents, the dynamics of the system bifurcate around the threshold \( Z^*_\lambda \), so that there are two regimes that may govern the dynamics of the system. As \( u_t \) crossed the equilibrium threshold from left to right, \( \lambda_t \) starts to increase. Initially, \( \lambda_t \) grows approximately exponentially. As more and more agents start to speculate, the rate at which \( \lambda_t \) is growing falls, proportionally to the declining fraction of agents that still have positions in the weak currency. When \( u_t \) crosses the equilibrium threshold from right to left, \( \lambda_t \) declines in a similar fashion. Hence, depending on the regime, the evolution of \( \lambda_t \) is governed by either

\[
\dot{\lambda}_t^\downarrow = -\zeta \lambda_t, \quad \text{or} \quad \dot{\lambda}_t^\uparrow = \zeta (1 - \lambda_t),
\]

where \( \dot{\lambda}_t^\downarrow \) denotes the time derivative of \( \lambda_t \) to the left of the threshold, and \( \dot{\lambda}_t^\uparrow \) that to the right. An upward bifurcation of the dynamical system is a situation in which the evolution of \( \lambda_t \) is described by \( \dot{\lambda}_t^\uparrow \) until \( \lambda_t \) almost reaches 1. A downward bifurcation is defined analogously.\(^4\)

Now consider the behaviour of \( \lambda_t \) conditional on the event that no large jump arrives until \( \lambda_t \) reaches some small neighbourhood of 0 or 1. As shown in Burdzy et al. [5, theorem 2], close to the equilibrium threshold and for sufficiently large \( \zeta \), the probability of an upward bifurcation is approximately \( 1 - \lambda_t \), and the probability of a downward bifurcation approximately \( \lambda_t \). Thus, around the threshold the system bifurcates almost surely, at a ratio which depends inversely on the proportion of agents currently locked into either action. This bifurcation result depends only on the fact that the dynamical system is of the form in figure 3, so applies to our setting.

A consequence of the result is that when \( \zeta \) becomes large, the equilibrium must converge to a curve that lies everywhere to the left of the line \( u_t = \ell \). Why? Suppose \( Z^*(\lambda_t) > \ell \) for some \( \lambda_t < 1 \). As \( \zeta \to \infty \), the system bifurcates upwards with positive probability \( 1 - \lambda_t \). Since \( u^* \) is downward sloping, under the induced dynamics, \( \lambda_t \) reaches the critical value determined by the curve \( u^* \) very quickly—refer to figure 1. Moreover, since the variance of \( W_t \) is small relative to the speed at which \( \lambda_t \) changes, the value of the fundamental \( u_t \) will vary very little over the short period it takes to reach \( u^* \). Furthermore, there is always a positive probability that the speculator will not receive another revision opportunity before \( u^* \) is reached. While the speed at which \( \lambda_t \) is changing depends on \( \zeta \), the probability of being locked into the current action when \( u^* \) is reached does not depend on \( \zeta \), because all speculators receive revision opportunities at rate \( \zeta \). If the speculator is still locked into her action when \( u^* \) is reached, she obtains a payoff of 1 if she chooses to speculate against the pegged currency, which—for small \( \zeta \)—dwarfs the costs incurred during the short period, and

\(^4\)This definition is still somewhat imprecise, because we have not specified what it means for \( \lambda_t \) to almost reach 0 or 1. A more precise definition of the concept of bifurcation, that takes care of this detail, emerges from the proof of theorem 3 below.
of 0 if she doesn’t. (The same argument still applies if we add the possibility of large shocks, since they only lead to positive jumps in \( u_t \).)

In the limit, as \( \zeta \to \infty \), a collapse of the peg is almost surely triggered by a disturbance due a large shock originating from the compound Poisson process, since \( W_t \) doesn’t move much relative to the agent’s lock-in time. So we now explicitly consider the influence of such shocks on the dynamical system depicted in figure 3. Suppose agent \( i \) makes a decision at \( t \). The waiting time between large shocks is exponentially distributed, so that the probability that the next large shock arrives before time \( t + \tau \) is given by \( 1 - \exp(-\vartheta \tau) \) (recall that \( \vartheta \) is the parameter of the compound Poisson process). Intuitively, at time \( t \), the relative probability weight that the next shock arrives in the small interval starting at \( t + \tau \) converges to \( \vartheta \exp(-\vartheta \tau) \) for \( \tau \to 0 \). Now, conditional on the event that a large shock arrives during this interval and the agent is still locked in at time \( t + \tau \), for any value \( \zeta \) and any joint strategy characterised by a continuous and downward sloping curve \( Z \), the probability that the fundamental will cross \( u^* \) during the agent’s lock-in time after a large shock hits the dynamical system is a time-homogeneous function \( \rho(\lambda_t, u_t; Z) \). Given the shape of the curve \( u^* \), the function \( \rho(\cdot) \) must be continuous, increasing in its first two arguments, and positive.

Let \( Z(\bar{u}) \) denote the “degenerate” joint strategy where speculators speculate against the weak currency if and only if \( u_t \geq \bar{u} \) independent of \( \lambda_t \). First, observe that:

**Lemma 2.** The equation \( \vartheta \int_0^1 \rho(\lambda, \bar{u}; Z(\bar{u})) \, d\lambda = r \) has a unique solution for \( \bar{u} \).

Denote this solution by \( \bar{u} \). Now consider the function:

\[
V(u_t) := \vartheta \int_0^1 \rho(\lambda, u_t; Z(u_t)) \, d\lambda.
\]

(This function closely resembles the equation given in Frankel and Pauzner [12, p. 297]). We will say that speculation is a risk-dominant action at \( u_t \) if \( V(u_t) \geq r \), and that refraining from speculation is a risk dominant action at \( u_t \) if \( V(u_t) \leq r \). The following result then characterises the equilibrium of the model for \( \zeta \to \infty \).

**Theorem 3.** As \( \zeta \to \infty \), the equilibrium threshold \( Z_{\zeta}^* \) converges to the threshold \( Z \) characterised by the condition \( V(Z(\lambda_t)) = r \). Therefore, as \( \zeta \to \infty \), agents always choose a risk-dominant action.

---

5The function must be time-homogeneous, given that directly after a shock that takes the dynamical system to \( (\lambda_t, u_t) \), the expected future dynamics of the system depend only on the new value of \( u_t \) and on \( \lambda_t \). This is because the Poisson revision process, as well as \( W_t \) and \( J_t \) are stochastic processes without memory. In general no closed form expression may exist for the function \( \rho \), though conceivably it may be approximated by numerical simulation.
In the proof of the theorem we show that, for large $\zeta$, at every point $u_t$ lying on the curve $Z^*_\zeta(\lambda_t)$, we must approximately have $V(u_t) = r$. Clearly, to satisfy this requirement $Z^*_\zeta$ must be close to a vertical line and, as $\zeta \to \infty$, $Z^*_\zeta$ must therefore converge to the vertical line $Z(\bar{\pi})$ of lemma 2—see figure 5.

Theorem 3 allows us to compare the equilibrium of this dynamic currency crisis model to that of the static global game set-up of Morris and Shin [21] where agents receive signals about $u_t$. In the static global game model, as noise vanishes but strategic uncertainty remains, agents choose the action that gives the highest payoff as if they are completely unsure about the choices of other agents and believe that the fraction of speculators that attack the currency can be anything from 0 to 1. Equation (4) expresses a rather similar idea. The value $V(u_t)$ describes the expected payoff of an agent who receives a revision opportunity when the fundamental equals $u_t$ and chooses to attack, but is unsure about how many agents will eventually join the attack. This value of $V(u_t)$ is determined by taking the average value of the payoff, $\vartheta \rho(\cdot) / \zeta$, ranging over all possible values for $\lambda_t$, and comparing this with the expected costs associated with speculation, $r / \zeta$. The equilibrium relation between $\vartheta \rho(\cdot)$ and $r$ remains intact as $\zeta$ tends to infinity.

The condition that $V(u_t) = r$ at the limiting equilibrium threshold immediately yields an intuitive comparative static result with respect to the rate of return $r$. If $r$ is increased, then the threshold will shift to the right, which indicates that the policy maker can reduce speculative pressure on the currency by increasing the interest rate differential—a result that conforms to intuition.\(^6\) Note that the Poisson distributed large shocks to $u_t$ are crucial for this comparative static result not to vanish in the limit.

To appreciate the subtle role played by the large shocks in this paper’s dynamic model, it is instructive to compare the comparative static results above to those obtained by Guimarães

\(^6\)This is only a partial comparative statics result, because in the current model the interest rate set by the policy maker is not determined endogenously. This issue is investigated in Daniëls et al. [8] in a simpler setting.
In the model of Guimarães, $u_t$ is driven solely by a Wiener process, so that there is no risk of large shocks hitting the fundamental. Guimarães shows that if the limit is $\zeta \to \infty$ is taken, the threshold $Z^*$ converges to a vertical line at $\ell$, and from this he concludes that in the case of vanishingly small frictions his model takes on the property of the first generation model that a speculative attack takes place exactly when it leads to an immediate collapse of the exchange rate. In our view, however, studying solely the (Wiener induced) Brownian motion inside a currency crisis version of the model of Burdzy et al. [6] and Frankel and Pauzner [12] gives misleading results. This is because in such models the costs associated speculation are necessarily proportional to the agents’ lock-in time, so of order $\zeta$, while for any fixed horizontal distance between $u_t$ and $u^*$, the probability of the stochastic process crossing the threshold declines exponentially in $\zeta$ (see e.g. [16, section 13.4, theorem 5]). Because of this property, for any $\epsilon > 0$, however small, and large enough $\zeta$, the opportunity cost associated with speculation will always dominate the payoffs associated with it to the left of the line $\ell$, and never to the right of it, suggesting that interest rates do not influence the decisions of agents in the limit. In this case it is not possible to derive meaningful comparative static results as $\zeta \to \infty$. Therefore, Guimarães stresses the importance of considerable frictions on financial markets to understand how currency crises develop (although he does not provide a concrete explanation of why these frictions might be substantial). The above results show that the absence of meaningful comparative statics is a mathematical artifact of considering Brownian motion. For the more general class of stochastic processes considered here, intuitive comparative statics can be obtained, even when frictions are small.

6. Concluding Remarks

We considered the issue of the timing of a currency crisis in a model that is inherently dynamic, but has a structure similar to second generation currency crises models. Concretely, we extended the results of Burdzy et al. [6] and Frankel and Pauzner [12] to the currency crisis setting. While in the articles by Burdzy et al. and Frankel and Pauzner agents engage directly in a coordination game, in the model in this paper agents have to forecast a future policy decision, where the behaviour of the policy maker is influenced by the combined speculative pressure of all agents. This situation retains a number of features of a coordination game, foremost the presence of strategic complementarities, but the element of strategic forecasting adds additional complexity to the dynamic model.

As indicated in the introduction, many of the properties of the model that we derived in this paper closely resemble those of static global game currency crisis models in the spirit of Morris and Shin [21]. In static global games, a noisy information structure forces each agent to adjust her strategy take into account a global class of games, which leads to a unique equilibrium prediction. In the model in this paper the noisy information structure of the
static global game is absent, but possibility of being locked into her action forces agents to take into account a global state space which includes all possible combinations of \( \lambda_t \) and \( u_t \). In this broader sense, the model is still a global game model. As discussed in section 3, the model can be solved by an “infection” argument, which is roughly equivalent to the argument that is applied in the static context. Similar to the static model, the dynamic model has a unique equilibrium in rationalisable strategies. This equilibrium takes the form of a threshold that (in the limit) may be characterised in terms of a risk dominance condition, which is similar to the characterisation results obtained for the static model in Ref. [23]. Thus, in more than one way, the model in this paper can be regarded as a dynamical counterpart of the static global game currency crisis model.

Proofs

Lemma 4. The limits \( \overline{H} \) and \( \underline{L} \) alluded to in the main text are well defined, Lipschitz continuous, decreasing functions. Agent choosing in the graph are of \( \overline{H} \) (or \( \underline{L} \)) are indifferent between speculating or not, if they believe that all others follow a strategy where they attack if \((u_t, \lambda_t)\) is to the right of \( \overline{H} \) (respectively, \( \underline{L} \)), and refrain from this if \((u_t, \lambda_t)\) is to its left.

Proof. Let \( \pi(u_t, \lambda_t; Z) \) denote the expected payoff for an individual speculator of choosing “attack” when the state of the system is \((u_t, \lambda_t)\), and when others follow a strategy where they attack if \((u_t, \lambda_t)\) is to the right of the function \( Z : [0, 1] \rightarrow \mathbb{R} \) (with \( Z \) Lipschitz continuous\(^7\)), and refrain from this if \((u_t, \lambda_t)\) is to its left. This expected payoff is composed of the expected costs associated with speculation \( r \), versus the probability that \( u_t \) will cross the critical threshold \( u^* \) during the agent’s lock in time if she chooses to attack at time \( t \). Results by Burdzy et al. [5] (concretely, their lemma 2) imply that the function \( \pi(u_t, \lambda_t; Z) \) is continuous for fixed \( Z \). The initial curve \( H_0 \) may then be found as the graph of the relation \( \pi(u_t, \lambda_t; Z) = 0 \), where \( Z \) is chosen to lie everywhere to the right of \( u^* \) (implying that no other speculators choose to attack the currency).

Note that \( u^* : [0, 1] \rightarrow \mathbb{R} \) is continuously differentiable on a compact set and is therefore Lipschitz continuous (Dudley [9], p. 188). Let \( c \) be its Lipschitz constant. We will show that \( H_0 \), regarded as a function from \([0, 1]\) into \( \mathbb{R} \), is also Lipschitz continuous with constant \( c \) (the argument is a variation on that of Frankel and Pauzner [12], p. 301–302). Consider two arbitrarily chosen distinct points \((u_t, \lambda_t)\) and \((u'_t, \lambda'_t)\) in the graph of \( H_0 \), and suppose without loss of generality that \( \lambda_t < \lambda'_t \) (the shape of \( u^* \) then implies that \( H_0(\lambda_t) > H_0(\lambda'_t) \)). We compare two paths of the process for \( u_t \), one starting at \( u_t \) and one starting at \( u'_t \) under identical realisations of the stochastic process in (2). Recall that the curve \( H_0 \) is derived

\(^7\)A function \( f : [0, 1] \rightarrow \mathbb{R} \) is Lipschitz continuous (or Lipschitz) if there exists a constant \( c \) such that \( |f(\lambda) - f(\lambda')| \leq c|\lambda - \lambda'| \), for all \( \lambda, \lambda' \in [0, 1] \).
under the belief that all other agents refrain from attacking, and note that under this joint strategy the difference between $\lambda_t \leq \lambda'_t$ on the two paths can only shrink. Now suppose:

\[(5) \quad H_0(\lambda) - H_0(\lambda') > u^*(\lambda) - u^*(\lambda'), \quad \text{(where } u^*(\lambda) - u^*(\lambda') \leq c(\lambda' - \lambda), \text{ as } u^* \text{ is Lipschitz).} \]

Then surely, if a path starting from $(u'_t, \lambda'_t)$ has crossed $u^*$ at some time $t^* > t$ then the corresponding path from $(u_t, \lambda_t)$ must already have crossed $u^*$ before time $t^*$. But the only way an agent choosing at time $t$ can be indifferent both at $(u_t, \lambda_t)$ and $(u'_t, \lambda'_t)$ is if for every path of $u$ that has crossed $u^*$ at time $t^* > t$, the corresponding path starting from $(u_t, \lambda_t)$, has also crossed $u^*$ at time $t^*$. So the first inequality in (5) cannot hold, and we may conclude $H_0(\lambda) - H_0(\lambda') \leq c(\lambda' - \lambda)$. Therefore $H_0$ is Lipschitz with constant $c$.

For each $n \in \mathbb{N}$, $n > 0$, the curve $H_n$ may be found recursively as the graph of the relation

$$\pi(u_t, \lambda_t, H_{n-1}) = 0.$$ 

Following the reasoning by Frankel and Pauzner [12, p. 301–302], each $H_n$, regarded as a function, is again Lipschitz with constant $c$. The collection of all the $H_n$s is bounded (because of the existence of dominance regions) so converges pointwise to some limit $\overline{H}$. Moreover, since each $H_n$ has the same Lipschitz constant, the collection of the $H_n$s is equicontinuous, which implies that the $H_n$s in fact converge uniformly to $\overline{H}$, and that $\overline{H}$ has Lipschitz constant $c$, and a fortiori (Dudley [9], p. 51–53). Finally, the operation of constructing $H_{n+1}$ from $H_n$ is continuous in leftward or rightward shifts of the curve $H_n$. Since continuous operations preserve limits, we may conclude that the graph of $\overline{H}$ coincides with the relation $\pi(u_t, \lambda_t; \overline{H}) = 0$. The same property may be proved of the curve $\overline{L}$, by analogous arguments.

\[\blacksquare\]

**Proof of Theorem 3.** In Burdzy et al. [5], the mathematical companion paper to Frankel and Pauzner [12], the authors study the properties of a dynamical system governed by the equations in (3) when the variance of the stochastic process $W_t$ becomes small and approaches 0, and the jump component $J_t$ is absent (also in the model considered in [12], there is no compound Poisson process $J_t$). We can apply their theorem 2 by (a) choosing a rescaled unit of time (just like Frankel and Pauzner do in their proof of [12, theorem 3]), so that instead of taking $\zeta \to \infty$, the variance of $W_t$ shrinks to 0 and then (b) incorporating the possibility of a discrete jump directly in our argument.

To this end, we define a new time unit $\tilde{t} = t/\zeta$. In the new time units, the Poisson parameter for the revision process equals 1, the variance of $\tilde{W}_t$ is $1/\zeta$, the rate of arrival of the large shocks is $\tilde{\vartheta} = \vartheta/\zeta$, and costs of speculation are given by $\tilde{r} = r/\zeta$ per unit of time.

Let $Z : [0, 1] \to \mathbb{R}$ be a continuously differentiable, decreasing, Lipschitz continuous function and suppose that agents follow a strategy where they attack if $(u_t, \lambda_t)$ is to the right of the graph of $Z$ and refrain from doing so if $(u_t, \lambda_t)$ is to its left. By theorem 2 in Burdzy et al. [5] (see also Burdzy et al. [6]), as the variance of $\tilde{W}_t$ shrinks to 0, for any $\delta > 0$, the
and $\tau > 0$ and at any $(u_{t_0}, \lambda_{t_0})$ in the graph of $Z$, the probability that the process $\lambda_t$ bifurcates upwards before $\tau$ units of time have passed and subsequently reaches $1 - \delta$ approaches $1 - \lambda_{t_0}$. Similarly, the probability that the process $\lambda_t$ bifurcates downwards before $\tau$ units of time have passed and subsequently reaches $\delta$ approaches the value of $\lambda_{t_0}$. The difference between Burdzy et al. [5] and the model in this article is that these authors do not consider the possibility of large shocks, so that in our setting $\delta$ is reached conditional on the fact that no large shock arrives. This means that in our setting, the bifurcation results in [5] still apply but become statements of the form: with probability $\lambda_{t}$, the process for $\lambda_t$ is described by $\lambda_t^{\uparrow}$ until either $1 - \delta$ is reached or a large shock arrives; similarly for a downward bifurcation.

Now, for large $\zeta$ (or equivalently, small $\tilde{\theta}$), we will approximate expected payoffs of an agent choosing at time $t$ by supposing that during agent $i$’s lock-in time, the compound Poisson process $J_t$ jumps at most once. (As $\zeta \to \infty$, the probability that $J_t$ jumps twice during an agent’s lock-in period becomes negligible, as this probability is $O\left(\frac{1}{\zeta^2}\right)$ as $\zeta \to \infty$.8

The probability that a single jump arrives is of $O\left(\frac{1}{\zeta}\right)$ as $\zeta \to \infty$ but so are a speculator’s opportunity costs $r/\zeta$. Hence the probability that a jump arrives stays relevant.) The probability a shock arrives in the infinitesimal interval $dt$ is then given by $\tilde{\theta} e^{-\tilde{\theta}t}$ in the new time unit. If—and only if, given the argument in the main text, section 5—$J_t$ indeed jumps during an agent’s lock-in time, it is possible that $u_t$ crosses $u^*$ at some time at which she is still locked into her action. Given that all stochastic processes considered in the model are memory-less, the ex ante probability of $u_t$ crossing $u^*$ should depend solely on the position of $u_t$ and $\lambda_t$ just before the time of the jump, and is given by a time-invariant function $\rho(\cdot)$ that depends on $u_t$ and $\lambda_t$ just before the time of the jump.

Recall that each $Z_{\zeta}^*$ is Lipschitz continuous but not necessarily continuously differentiable. Because theorem 2 in Burdzy et al. [5] is stated for continuously differentiable functions $Z$, we will approximate each $Z_{\zeta}^*$ by smooth functions $Z_{\zeta}^n$. Fortunately, Lipschitz functions with constant $c$ can be approximated uniformly by functions that are infinitely differentiable (so-called $C^\infty$-functions) with the same Lipschitz constant $c$. One way to do this is through a technique known as “mollification”; see e.g. Azagra et alia [3, p. 1370–1371] for details. Fixing $\zeta$, lemma 2(iii) in [5] shows that as $Z_{\zeta}^n \to Z_{\zeta}^*$ the dynamics of $(u_t, \lambda_t)$ induced by each $Z_{\zeta}^n$ converge to those induced by $Z_{\zeta}^*$. This means that for each $\zeta$ we may choose $n$ such that for each point on the graph of $Z_{\zeta}^*$ the expected difference between attacking and refraining under the dynamics induced by $Z_{\zeta}^n$ is within $\frac{1}{\zeta^2}$ of the expected difference between attacking and refraining under the dynamics induced by $Z_{\zeta}^*$

8Recall that function $f(\zeta, \cdot)$ is called “$O\left(g(\zeta)\right)$ as $\zeta \to \infty$” if there are $\zeta_0$ and positive constant $k$ such that $|f(x)| \leq k|g(x)|$ for all $\zeta > \zeta_0$. In equations (6), (7) and (8), the notation “$O\left(g(\zeta)\right)$” refers to an error term that is of this order.
Normalise \( \tilde{t} = 0 \) and choose \( \lambda_0 \in (0, 1) \). Let \( u_0 = Z^*_\rho(\lambda_t) \). As shown in [12, proof of theorem 2], for large enough \( \zeta \) and \( n \), in equilibrium the expected payoff of choosing to speculate at \( (u_0, \lambda_0) \in Z^*_\rho \) can be approximated by the following equation:

\[
(1 - \lambda) \int_0^\infty \tilde{t} e^{-(\tilde{t}+1)i} \rho(\lambda^1_t, u_0; Z^*_\rho) d\tilde{t} + \lambda \int_0^\infty \tilde{t} e^{-(\tilde{t}+1)i} \rho(\lambda^1_t, u_0; Z^*_\rho) d\tilde{t} + O\left(\frac{1}{\zeta^2}\right)
\]

which may be rewritten as:

\[
\frac{\dot{\vartheta}}{\zeta} \left[ (1 - \lambda) \int_0^\infty e^{-(\tilde{t}+1)i} \rho(\lambda^1_t, u_0; Z^*_\rho) d\tilde{t} + \lambda \int_0^\infty e^{-(\tilde{t}+1)i} \rho(\lambda^1_t, u_0; Z^*_\rho) d\tilde{t} \right] + O\left(\frac{1}{\zeta^2}\right),
\]

where \( \lambda^1_t = \lambda_0 \exp(-\zeta \tilde{t}) \) and \( \lambda^2_t = 1 - (1 - \lambda_0) \exp(-\zeta \tilde{t}) \). Here, the part between brackets in expression (7) corresponds exactly to the equation in [12, proof of theorem 2]. The authors show by change of variables that this part approximately equals the expression:

\[
\int_1^0 w(l) \left[ \rho(l, u_0; Z^*_\rho) \right] dl,
\]

with weights \( w(l) \) which are given by:

\[
w(l) = \begin{cases} \left( \frac{l}{\lambda_0} \right) \dot{\vartheta} & \text{if } l \leq \lambda_0 \\ \left( \frac{1-l}{1-\lambda_0} \right) \dot{\vartheta} & \text{if } l > \lambda_0 \end{cases}
\]

As \( \dot{\vartheta} = \vartheta/\zeta \to 0 \), the weights converge to 1. If \( Z^*_\rho \) is the equilibrium threshold, it must be that an agent who receives a revision opportunity in the graph of \( Z^*_\rho \) is indifferent. The opportunity costs associated with speculation are given by \( r/\zeta \), so that for large \( \zeta \) the indifference condition converges to:

\[
(8) \quad \vartheta \int_0^1 \rho(l, Z^*_\rho(\lambda_0); Z^*_\rho) dl + O\left(\frac{1}{\zeta}\right) = r,
\]

or, in other words, the returns \( r \) compensate for a “weighted” expected gross return from attacking at \( u_0 = Z^*_\rho(\lambda_0) \). Since \( \rho \) is increasing in \( u_t \), agents attack for sure to right of \( Z^*_\rho \) and refrain from attacking for sure to its left. It follows that for sufficiently large \( \zeta \) at every point \( u_t \) lying on the curve \( Z^*_\rho(\lambda_t) \), we must approximately have \( V(u_t) = r \). This condition corresponds to that in the statement of theorem 3. Clearly, to satisfy this requirement \( Z^*_\rho \) must be close to a vertical line and, as \( \zeta \to \infty \), \( Z^*_\rho \) must therefore converge to the vertical line \( Z(\bar{u}) \) of lemma 2. Agents attack to the right of \( Z(\bar{u}) \) (i.e., when \( V(u_t) > r \)), and refrain from attacking its left (\( V(u_t) < r \)). ■
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