A Reputation-Based Theory of Spatially-Separated Duopoly Competition and Bargaining

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Submitted: June 27, 2009.
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*I am indebted to my adviser David Pearce for his continuous guidance, support and encouragement. I am grateful to Ennio Stacchetti for many discussions which resulted in significant improvements of the paper. I would also like to thank Alessandro Lizzeri, Tomasz Sadzik and Ariel Rubinstein for helpful comments and suggestions. All the remaining errors are my own.

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1 Introduction

This paper develops a reputation-based model to investigate and highlight the influence of “posted prices” and “bargaining postures” on spatially-separated duopoly competition. I consider the following simple market set-up: There are two spatially separated stores selling an indivisible homogeneous good to a single buyer who wants to consume only one unit. Locations of the stores are fixed at all times and the valuation of the good is one for the buyer, zero for the sellers. All players are impatient and there is no informational asymmetry regarding the players’ valuations and time preferences. The buyer can negotiate with one potential seller at a time, but with some delay, he can move back and forth between the two sellers.

The resulting multilateral bargaining problem has a continuum of subgame perfect equilibria; this set depends on the fine details of the bargaining protocol.\footnote{For related literature see Osborne and Rubinstein (1990) and the references therein.} By contrast, this paper shows that a slight perturbation of the problem by introducing “behavioral” types engenders an essentially unique equilibrium. Following Kreps and Wilson (1982) and Milgrom and Roberts (1982), I assume that each of three players suspects that the opponents might have some kind of behavioral commitment forcing them to insist on a specific allocation. Posted prices, for example, can naturally give rise to the fear that the sellers might not be willing to negotiate and would insist on the price they post. Even if players assign small probabilities to the behavioral types, the profusion of equilibria of the multilateral bargaining problem reduce to a unique one.

I also show that the unique equilibrium allocation does not depend on the fine details of the bargaining protocols, nor do the sellers extract all the surplus of the buyer because of the positive travel cost. Instead, it depends on the behavioral demands (posted prices) and initial reputations (the probability of being the behavioral type) as well as the time preferences of the players.

Behavioral types take an extremely simple form. Parallel to Myerson (1991), Kambe (1999), Abreu and Gul (2000), and Abreu and Pearce (2003), a behavioral player always demands a particular share and accepts an offer if and only if it weakly exceeds that share. A behavioral seller, for example, never offers a price below his original posted price, and never accepts an offer below that price. Similarly, a behavioral buyer always offers a
particular amount, and will never agree to pay more. Thus, the buyer must choose either to mimic this obstinate type, or reveal his rationality and continue negotiation with no uncertainty regarding his actual type.

In addition, I assume for simplicity that the timing and location decisions of the behavioral buyer are the same as those of his rational counterpart. In this paper, for tractability purposes I limit attention to the case of a single behavioral type of each player. In future work I plan to let the players choose which of many behavioral types to imitate. In that framework, the buyer’s reputation may be updated in subtle ways by his timing decisions.

In the presence of the behavioral types, the equilibrium outcomes of the discrete-time bargaining problem à la Rubinstein converge to a unique limit, independent of the fine details of the bargaining protocols, as players can make increasingly frequent offers. Moreover, this limit is the unique equilibrium outcome of the following continuous-time (war of attrition) problem. Before the negotiation starts, all players know the behavioral demands of their opponents (i.e. the buyer knows the posted price in each store), and the buyer initially decides which store to go to first. Upon arrival at a store, the buyer and the seller start to play the concession game. At any given time, a player either accepts his opponent’s behavioral demand or waits for a concession. At the same time, the buyer decides whether to stay or to leave the store.

The equilibrium of the continuous-time bargaining problem is unique up to the buyer’s selection of store to visit first. In equilibrium, the buyer does not visit a given store more than once as long as the sellers’ posted prices (behavioral demands) are the same. Thus, in equilibrium the buyer enters store 1, for example, at time 0, and starts playing the concession game with the seller until a specific finite time. If neither player concedes to his opponent, the buyer leaves store 1 at this time with a higher reputation, and goes directly to store 2 to continue the concession game with the second seller. The negotiation in the second store ends at some finite time with certainty, at which point the players’ reputations simultaneously reach 1.

The departure time of the buyer from store 1 is an increasing function of the distance

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2Technically, this means the behavioral buyer is an “endogenous” type in the sense of Abreu and Pearce (2003).
between the stores and the reputation of the second seller. Furthermore, the buyer’s expected payoff in store 1 is a decreasing function of his departure time from store 1, while his expected payoff in store 2 is an increasing function of it. The equilibrium strategy of a player in the concession game is some continuous and strictly increasing distribution function with a constant hazard rate. That is, each player concedes by choosing the timing of acceptance randomly with a constant (instantaneous) acceptance rate.

When the sellers’ posted prices are different and the distance between the stores is short relative to the difference between the posted prices, the structure of the equilibrium strategy dramatically changes. In this case, the buyer never negotiates with the seller whose posted price is higher, though he may prefer to visit this store at time 0 in order to make the “take it or leave it” ultimatum. However, if the distance between the stores is large relative to the difference between the posted prices, then the buyer may bargain with the seller that has the higher posted price, and the negotiation with this seller may take a significant amount of time. In this case, if the buyer has a high initial reputation relative to the sellers, then the buyer chooses the seller with the higher posted price to visit first at time 0.

The market structure in this paper is different from those in Binmore and Herrero (1988), Riley and Zeckhauser (1983), Rubinstein and Wolinsky (1985), and many others because they assume a Walrasian market where there are an infinite number of sellers and buyers, and each player waits to be matched (with an exogenous matching mechanism) to a bargaining partner. In contrast, in my model there is only one buyer and two sellers, and the buyer endogenously decides where to go, and when. Moreover, the only source of uncertainty is about the players’ strategic intent or strategic postures. Also, unlike Bester (1988, 1989 and 1993) and Shaked and Sutton (1984), locations of the stores are fixed at all times, and the players are able to make their offers “frequently”.

Section 2 and 3 analyze the continuous-time bargaining problem where the strategies of the players are effectively either to stick with the initial behavioral demand or accept the opponent’s behavioral demand. Section 2 deals with the case in which the behavioral demands of the sellers are equal. Section 3 investigates the case with dissimilar demands. Section 4 explores the convergence of discrete-time bargaining problem to the “protocol-
free” continuous-time problem. Finally, Section 5 makes some closing remarks.

2 The Continuous-Time Bargaining Problem

Consider the following market environment: There are two spatially-separated stores selling an indivisible homogeneous good to a single buyer who wants to consume only one unit. The sellers are located at opposite ends of a straight line while the buyer’s position is midway in between the two stores. The sellers’ locations are fixed at all times. To purchase the good, the buyer has to visit a seller’s store. However, moving from one store to another takes time. The valuation of the good is one for the buyer and zero for the sellers. There is no informational asymmetry regarding the valuations of the good.

Assume also that there is some reason to suspect that each player might have some kind of irrational commitment forcing him to insist on a specific allocation which is known by all players before the game starts. To be more specific, suppose that there is (possibly) a small but strictly positive probability, \( z_i \), that seller \( i \) is a behavioral type implementing the following strategy: He always offers \( \alpha_i \in (0, 1) \), rejects any price offer strictly below it and accepts any price offer weakly above it.

Similarly, suppose that there is a small but strictly positive probability, \( z_b \), that the buyer is a behavioral type executing the following strategy: He always offers \( \alpha_b \in (0, 1) \) to sellers, accepts any price offer less than or equal to \( \alpha_b \) and rejects any price offer strictly above it. The timing and location decisions of the behavioral type buyer are the same as those of the rational buyer. In this sense, the behavioral type for the buyer is endogenous.\(^3\) In this section, I assume that the sellers’ behavioral demands are the same and equal to \( \alpha \), and that they are incompatible with the buyer’s behavioral demand, i.e. \( \alpha_b < \alpha \).

The continuous-time bargaining problem between the buyer and the sellers is modeled in the following way. Initially, the buyer decides which store to go to first. Upon arrival at store \( i \in \{1, 2\} \) at time 0, the buyer and seller \( i \) instantaneously begin to play the following concession game: At any given time, a player either accepts his opponent’s behavioral demand or waits for a concession. At the same time, the buyer decides whether to stay

\(^3\)See footnote 2
or leave store $i$. Concession of the buyer or seller $i$, while the buyer is in the store, marks the completion of the game. In case of simultaneous concession, surplus is split equally.\footnote{This particular assumption is not crucial because simultaneous concession occurs with probability zero in equilibrium.} If the buyer leaves store $i$ and goes to store $j$, the buyer and seller $j$ start playing the concession game upon the buyer’s arrival at that store. Both sellers can perfectly observe the buyer’s moves throughout the game. Thus, the players’ actual types are the only source of uncertainty in the game. After leaving store $i$ and traveling part way to store $j$, the buyer could, if he wished, turn back and enter store $i$ again. I denote this continuous-time bargaining problem by $G$.

I propose to model the duopoly competition, where the buyer searches and negotiates in the hope of finding a better price while the sellers compete over the buyer, using the continuous-time bargaining problem $G$. It is important to note that $G$ is a modified war of attrition game between two sellers and the buyer, and this modeling selection is not arbitrary. Section 4 shows that equilibrium outcomes of a discrete-time bargaining game à la Rubinstein converge to a unique limit, independent of the exogenously given bargaining protocols as long as players can make frequent offers, and this limit is equivalent to the unique outcome of the continuous-time bargaining problem $G$.

Equilibrium of the continuous-time bargaining problem is unique up to the buyer’s selection of store to visit first at time 0. A short descriptive summary of the equilibrium strategy is as follows. At time 0, the buyer enters store 1, for example, and starts playing the concession game with the seller until time $T_{d1}$. At this time the buyer leaves store 1 and goes directly to store 2. Once the buyer arrives at store 2, the buyer and seller 2 play the concession game until $T_{e2}$, when both players’ reputations reach 1. That is, by time $T_{e2}$ the game ends with certainty if one of the players is rational. So, in equilibrium the buyer visits each store at most once.\footnote{In this section, I assume that the sellers’ behavioral demands are the same.}

The departure time of the buyer from store 1, $T_{d1}$ is bigger than or equal to zero depending on the buyer’s initial reputation (probability that the buyer is the behavioral type), his time preference and behavioral demand relative to the sellers. However, in equilibrium the buyer may never leave store 1 if the traveling cost is sufficiently high or his behavioral demand is arbitrarily close to the sellers’ demand, making traveling to
haggle with the other seller not worth his time.

Each player’s equilibrium strategy in the concession game is a continuous and strictly increasing distribution function. That is, in equilibrium both the buyer and the sellers concede by choosing the timing of acceptance randomly with a constant hazard rate (or instantaneous acceptance rate). Therefore, at any moment the players are indifferent to either accepting the opponent’s behavioral demand or waiting. The hazard rate of a player depends only on the behavioral demands and his opponent’s time preferences.

For the sake of simplicity in presentation and notation, I will focus for the moment on strategies where the buyer visits each store at most once. Appendix A will consider more elaborate strategies to prove that all the results in Section 2 hold without this restriction, which is a consequence of the equilibrium.

The buyer’s strategy in game G has two parts. The first part, \( \sigma_b \), determines the buyer’s location as a function of history. Assume, without loss of generality, in equilibrium the buyer visits store 1 first and then store 2. Let \( T_1^d \) denote the time that the buyer leaves store 1 if no agreement has been reached yet. Denote by \( \omega_i \) the time that the buyer starts negotiating with seller \( i \) (if agreement has not been reached yet). That is, \( \omega_1 = 0 \) and \( \omega_2 = T_1^d + \Delta \) where \( \Delta \) is the travel time between the stores. For notational simplicity, I manipulate the subsequent notation and denote \( \omega_2 \) by 0. That is, I reset the clock once the buyer arrives in store 2 (but not the players’ reputations).\(^6\)

The second part is a pair of right continuous distribution functions \( F_i^b : \mathbb{R}_+ \cup \infty \rightarrow [0, 1], i = 1, 2. \)\(^7\) Thus, for each \( t \), \( F_i^b(t) \) is the probability that the buyer concedes to seller \( i \) by time \( t \) (inclusive). Similarly, seller \( i \)'s strategy in the game G is a right continuous distribution function \( F_i : \mathbb{R}_+ \cup \infty \rightarrow [0, 1] \) such that for all \( t \geq 0 \), \( F_i(t) \) denotes the probability that seller \( i \) concedes to the buyer by time \( t \) (inclusive).

Given the strategy of the buyer, let \( z_b(t) \) denote the buyer’s reputation (probability that the buyer is the behavioral type) at time \( t \geq 0 \). It is updated according to the Bayes’ rule and is consistent with the buyer’s strategy: For example, since the buyer visits seller

\(^6\)Thus, with some manipulation of the notation, I define each player’s distribution function as if the concession game in each store starts at time 0.

\(^7\)Since the buyer leaves store 1 at time \( T_1^d \), \( F_1^b(\cdot) \) is defined over \( [0, T_1^d] \) corresponding to the time frame that the buyer is in store 1 according to \( \sigma_b \).
1 first, for any \( t \geq 0 \), we have \( z_b(t) = z_b / (1 - F_b^1(t)) \), and \( z_b(t) \) is no less than the buyer’s initial reputation \( z_b \). Furthermore, since the buyer leaves store 1 at time \( T_1^d \), it must be that \( F_b^1(T_1^d) \leq 1 - z_b \). On the other hand, the buyer visits store 2 if the players cannot reach an agreement in store 1, implying that \( F_b^2(T_2^d) \leq 1 - z_b(T_1^d) \) where \( T_2^d \) denotes the time that the continuous-time bargaining problem ends in store 2. Since I consider strategies where the buyer visits the stores at most once, it must be that \( F_1(T_1^d) \leq 1 - z_1 \) and \( F_2(T_2^d) \leq 1 - z_2 \).

Given \( F_b^i \), seller \( i \)’s expected payoff of conceding to the buyer at time \( t \geq 0 \) is

\[
U_b(t, F_b^i) := \alpha \int_0^t e^{-rsy} dF_b^i(y) + \frac{1}{2} (\alpha + \alpha_b)[F_b^i(t) - F_b^i(t^-)]e^{-rt} + \alpha_b[1 - F_b^i(t)]e^{-rt}
\]

with \( F_b^i(t^-) = \lim_{y \uparrow t} F_b^i(y) \).

In a similar manner, given \( F_i \), the expected payoff of the buyer who concedes to seller \( i \) at time \( t \geq 0 \) (conditional on not reaching a deal with seller \( j \) if store \( j \) was visited first) is

\[
U_b(t, F_b^i) := (1 - \alpha_b) \int_0^t e^{-rsy} dF_i(y) + \frac{1}{2} (2 - \alpha + \alpha_b)[F_i(t) - F_i(t^-)]e^{-rt} + (1 - \alpha)[1 - F_i(t)]e^{-rt}
\]

where \( F_i(t^-) = \lim_{y \uparrow t} F_i(y) \).

Define the strategy profile of the continuous-time bargaining problem \( G \), \( \sigma^* = (F^*; \sigma_b^*) \) where \( F^* = (F_1, F_2, F_b^1, F_b^2) \), in the following way. For all \( i \in \{1, 2\} \) and \( t \geq 0 \),

\[
F_i(t) = 1 - c_i e^{-\lambda t} \quad \text{and} \quad F_b^i(t) = 1 - c_b^i e^{-\lambda_i^t}
\]

where

\[
\lambda = \frac{(1 - \alpha)r_b}{\alpha - \alpha_b} \quad \text{and} \quad \lambda_i = \frac{\alpha_b r_i}{\alpha - \alpha_b}.
\]

Suppose without loss of generality that the buyer arrives at store 1 at time 0 and leaves store 1 at time \( T_1^d \) to go to store 2. Then

\[
c_2 = z_2 e^{\lambda T_2^d} \quad \text{and} \quad c_b^2 = z_b(T_1^d) e^{\lambda_b^2 T_2^d}
\]

\( ^8 \)In equilibrium, it must be that \( F_1(T_1^d) = 1 - z_1 \), and given that the buyer visits store 2 \( F_2(T_2^d) = 1 - z_2 \) and \( F_b^2(T_2^d) = 1 - z_b(T_1^d) \).

\( ^9 \)\( U_i \) is evaluated at time 0 in “real time”.

\( ^{10} \)If the buyer visits seller \( i \) first, then \( U_b^i \) is evaluated at time 0 (in real time). Otherwise, it is evaluated at time \( w_i + \Delta \) (in real time).
where
\[
T^e_2 = \min\{-\frac{\log z_2}{\lambda}, -\frac{\log z_b(T^d_d)}{\lambda_b}\}
\]
and
\[z_b(T^d_1) = \frac{z_b}{1 - F_b(T^d_1)}.\]

Notice that the strategy \(\sigma^*\) does not specify the buyer’s choice of store to visit at time 0. However, for each \(i \in \{1, 2\}\), \(\sigma^*\) prescribes the player’s strategies (except the parameters \(c_i \in [z_i, 1], c^b_i \in [z_b, 1]\) and \(T^d_i \geq 0\)) in the subgame following the buyer’s arrival at store \(i\).

**Proposition 2.1.** Let \(\sigma\) be a strategy profile of the continuous-time bargaining problem \(G\) in which the buyer visits each store at most once. If \(\sigma\) is a sequential equilibrium of \(G\), then \(\sigma \equiv \sigma^*.\)

I defer the proofs of all the results in this section to Appendix A.

Proposition 2.1 does not fully characterize the equilibrium strategy. It rather indicates certain properties that the equilibrium strategy must satisfy. Also, it is silent regarding the strategies where the buyer can visit stores multiple times.

Proof of Proposition 2.1 uses arguments in Hendricks, Weiss and Wilson (1988) and is analogous to the proof of Lemma 1 in Abreu and Gul (2000). In an equilibrium, if a player’s strategy has a discontinuity point at some time \(t\), his opponent prefers to wait a little longer, instead of conceding in some \(\epsilon\)-neighborhood of \(t\). Therefore, if \((F_i, F^b_i)\) are equilibrium strategies in the interval \([0, T]\) where \(T\) is either equal to \(T^d_i\) or \(T^e_i\) depending on which store the buyer visits first, then there cannot be common discontinuity point for these distribution functions on this interval.

On the other hand, if a player does not concede to his opponent during the time interval \([t, t'] \subset [0, T]\), his opponent prefers to wait in the interval \([t, t' + \epsilon]\) for some small but positive \(\epsilon\). Along with the previous argument, in equilibrium a player’s strategy cannot have a discontinuity point in \((0, T]\). Therefore, equilibrium strategies \((F_i, F^b_i)\) are strictly increasing, continuous and differentiable over \((0, T]\), implying that players are indifferent between conceding and waiting at any time of the concession game.\(^{12}\) A simple

\(^{11}\)I remark that \(\sigma\) is equivalent to \(\sigma^*\) up to the buyer’s choice of store to visit first and for each \(i \in \{1, 2\}\), the parameters \(c_i, c^b_i\) and \(T^d_i\) in the strategies following the subgame that the buyer arrives at store \(i\) at time 0.

\(^{12}\)Notice that \(F_i\) or \(F^b_i\) (not both) may be discontinuous at 0.
manipulation in the utility functions gives us the functional form of these distribution functions.

In equilibrium, each seller concedes to the buyer by choosing randomly the timing of acceptance with a constant hazard rate $\lambda$. Notice that the hazard rate of a seller depends only on the behavioral demands and the buyer’s time preferences. So, the sellers’ hazard rates are identical in equilibrium. On the other hand, the buyer’s hazard rate $\lambda^b_i$ may be different for each seller.

Proposition 2.1 indicates that in equilibrium, if the buyer visits store 1 first and later store 2, for example, then the haggling ends no later than the time $T^{e}_2$ with probability 1 if at least one of the players is rational. Since the buyer leaves store 1 at time $T^{d}_1$ (if the buyer cannot reach an agreement in store 1 by this time), his reputation at this departure time $z_b(T^{d}_1)$ is strictly higher than $z_b$ since $F^1_b$ is an increasing function.

In the equilibrium strategy where the buyer visits store 1 before store 2, let $\tau_2$ denote the time that seller 2’s reputation reaches 1 and let $\tau^2_b$ denote the time that the buyer’s reputation reaches 1. Therefore, if seller 2 does not concede to the buyer with positive probability at time 0, the time that the buyer enters store 2, then his reputation reaches 1 at time $\tau_2 = \frac{-\log z_b}{\lambda}$ since $F_2(\tau_2) = 1 - z_2$ must be satisfied. On the other hand, if the buyer does not concede to seller 2 at time 0 and if $z_b(T^{d}_1)$ (the buyer’s reputation when he enters store 2) is less than 1, then his reputation reaches 1 at time $\tau^2_b = \frac{-\log z_b(T^{d}_1)}{\lambda^b_2}$ because $F^2_b(\tau^2_b) = 1 - z_b(T^{d}_1)$. A rational player immediately accepts his opponent’s offer when his opponent’s reputation is 1 so the concession game in store 2 ends by the time $T^{e}_2 = \min\{\tau_2, \tau^2_b\}$.

Given that the game ends by the time $T^{e}_2$, simple algebra yields the values for $c_2$ and $c^2_b$. These two variables determine who makes the initial probabilistic acceptance in store 2. If $\tau^2_b < \tau_2$, the buyer builds his reputation faster than seller 2.\footnote{The buyer’s initial reputation or hazard rate must be higher.} In this case, the game will end by the time $\tau^2_b$. Therefore, seller 2 needs to adjust his strategy so that his reputation also reaches 1 by this time. The only way he can do this adjustment is by changing $F_2(0)$, the probability that seller 2 concedes to the buyer at the time that the buyer enters his store (initial probabilistic concession).

Note that in equilibrium, at most one player makes an initial probabilistic concession.
I call a player *strong* if he receives this probabilistic gift from his opponent and *weak* if he does not.

Let \( \delta \) denote the discount factor for the buyer that occurs due to the time required to travel from one store to the other.\(^{14}\) Notice that as this time, or equivalently the distance between the stores, decreases to zero, \( \delta \) converges to 1.

The next result determines the time that the buyer leaves the store he initially visits and the player who has to offer an initial probabilistic gift to his opponent.

**Proposition 2.2.** For \( \delta \) sufficiently close to 1, in the equilibrium strategy \( \sigma^* \) (where the buyer arrives at store 1 at time 0), the buyer is always the strong player in the concession game with the second seller. That is,

\[
c_2 = z_2 e^{\lambda T_2^d}, \quad c_b^2 = 1, \quad \text{where} \quad T_2^e = -\frac{\log z_b(T_1^d)}{\lambda_b^2}
\]

Moreover,

\[
c_1 = z_1 e^{\lambda T_1^d}, \quad c_b^1 = \begin{cases} 
z_b e^{\lambda_b T_1^d}, & \text{if } z_b < X_1 \\ 1, & \text{otherwise,} \end{cases}
\]

where the optimal time for the buyer to leave store 1 is

\[
T_1^d = \begin{cases} 
\min\{-\frac{\log z_1}{\lambda}, -\frac{\log(z_b/X_1)}{\lambda_b}\}, & \text{if } z_b < X_1 \\
0, & \text{otherwise,} \end{cases}
\]

where \( X_1 = \left(\frac{z_2}{A}\right)\frac{z_b^2}{A} \) and \( A = \frac{1-\alpha_b-\frac{1-\alpha}{\alpha}}{\alpha-\alpha_b} \).

In the equilibrium strategy \( \sigma^* \) where the buyer chooses store 1 to visit first, we can find the optimal departure time from this store as follows: First, note that during the concession game, seller 1 has two pure actions to choose; concede to the buyer or wait. However, the buyer has three pure actions; concede to seller 1, wait in store 1 or leave. Let \( \tau = \min\{\tau_1, \tau_b^1\} \) denote the time that the concession game would last if the buyer never leaves store 1. Proposition 2.1 implies that in equilibrium the buyer concedes with the constant hazard rate \( \lambda_1^1 \) to seller 1 so that at any time \( t \in [0, \tau] \) seller 1 is indifferent to either conceding or waiting. Likewise, seller 1 accepts the buyer’s offer with the constant hazard rate \( \lambda \) so that at any time \( t \in [0, \tau] \), the buyer is indifferent between conceding and waiting. If the buyer concedes to seller 1, his “instantaneous payoff” is \( 1 - \alpha \). We

\(^{14}\)That is, \( \delta = e^{-\tau \Delta} \).
can find seller 1’s hazard rate by equating \( 1 - \alpha \) to the buyer’s instantaneous payoff of waiting in store 1. Since the buyer is indifferent between conceding and waiting at all \( t \in [0, \tau] \), his expected payoff is equal to what he can achieve at time 0, i.e.

\[
v^1_b = F_1(0)(1 - \alpha_b) + (1 - F_1(0))(1 - \alpha)
\]  

(1)

On the other hand, the continuation payoff of the buyer in store 2, \( v^2_b(T^d_1) = F_2(0)(1 - \alpha_b) + (1 - F_2(0))(1 - \alpha) \), increases with \( T^d_1 \). If the buyer with reputation \( z_b(T^d_1) \) at time \( T^d_1 \) is the weak player in store 2, then it follows that the buyer’s continuation payoff in store 2 is \( 1 - \alpha \). Since the buyer’s instantaneous payoff in store 1 is also \( 1 - \alpha \), the buyer does not find it optimal to leave store 1 at time \( T^d_1 \), because \( \delta < 1 \).

Moreover, since the buyer’s reputation is an increasing function of the departure time, there must exist some value of \( T^d_1 \) such that the buyer is the strong player in store 2 with the reputation \( z_b(T^d_1) \). This implies that if the buyer leaves store 1 at this time, the concession game in store 2 ends at time \( T^d_2 = \frac{-\log z_b(T^d_1)}{\lambda^b_2} \), and the buyer’s continuation payoff in store 2 is

\[
v^2_b(T^d_1) = 1 - \alpha_b - z_2(\alpha - \alpha_b)[z_b(T^d_1)]^{\lambda^b_2/\lambda^b_2}
\]

Note that \( v^2_b(.) \) is a continuous and increasing function of the buyer’s reputation, \( z_b(T^d_1) \), and of departure time \( T^d_1 \) as long as \( z_b(T^d_1) \leq 1 \) holds. Therefore, it may come to a certain time such that the buyer is indifferent between conceding to seller 1 and leaving store 1 at this particular time.

Thus, optimality of the departure time and \( \sigma^* \) requires that at time \( T^d_1 \) the buyer must have built enough reputation to be the strong player in store 2 and \( 1 - \alpha \leq \delta v^2_b(T^d_1) \). In particular, if \( T^d_1 > 0 \) then its optimality implies that \( 1 - \alpha = \delta v^2_b(T^d_1) \). Solving this equality yields the equilibrium values of \( c_1, c^1_b \) and \( T^d_1 \), as well as \( X_1 \) and \( A \).

Proposition 2.2 implies that in the equilibrium strategy \( \sigma^* \) where the buyer chooses to visit store 1 first, the game finishes with probability 1 by time \( T^e_2 \) in store 2 if at least one of the players is rational and if the game does not end before this time. The buyer always picks the time to leave store 1, \( T^d_1 \), so that he is the strong player in store 2 with reputation \( z_b(T^d_1) \).

Moreover, the identity of the player who offers an initial probabilistic gift in store 1 depends on the time that the buyer leaves this store, i.e. \( T^d_1 \). The buyer is weak in
store 1 if the buyer’s initial reputation, $z_b$, satisfies $z_b \leq z_1^{\lambda_1 / \lambda} X_1$ or equivalently $T^d_1 = \tau_1$. In this case the buyer has to make an initial probabilistic concession with probability $1 - c_1^b$. For such values of $z_b$, the buyer is weak in store 1 because he cannot build enough reputation before time $\tau_1$—which is the time that seller 1’s reputation reaches 1 if he does not concede to the buyer at time 0—to force seller 1 for probabilistic concession at time 0.

On the other hand, the buyer is strong in store 1 whenever $z_b > z_1^{\lambda_1 / \lambda} X_1$ or $T^d_1 = -\log z_b / \lambda_1^{\lambda_1 / \lambda} X_1$. However, when $z_b \geq X_1$, the buyer is strong in store 2 even with his initial reputation $z_b$ and he prefers going to store 2 and playing the concession game with this seller over conceding to seller 1 at time 0. Thus, I call the buyer a distance-corrected strong player relative to seller $i$ if $z_b \geq X_j$ ($i, j \in \{1, 2\}$ and $i \neq j$).

In equilibrium, if the initial value of $z_b$ is large, in particular when the buyer is a distance-corrected strong player relative to seller 2, the buyer’s continuation payoff in store 2 is higher than his instantaneous payoff in store 1. Therefore, the buyer leaves store 1 immediately at time 0. Since the rational seller 1 knows that the buyer does not need to build reputation but rather plans to leave his store immediately, he accepts the buyer’s offer at time 0.

If the buyer is a strong but not a distance-corrected strong player relative to seller 2, then the buyer receives an initial probabilistic gift from seller 1, which is less than $1 - z_1$. That is, the concession game between the buyer and seller 1 lasts until the time of departure $T^d_1$, which is strictly positive in this case because the buyer needs to build his reputation in store 1 before it becomes optimal for him to go to store 2.

The existence of the second store gives the buyer a valuable opportunity to credibly threaten his opponent so that seller 1 has to offer a probabilistic gift at time 0. The buyer can force seller 1 to adjust his strategy and increase the amount of this initial gift by choosing the departure time earlier than $\tau_1$ (the time that seller 1’s reputation would reach 1 if the buyer could not leave his store). As the buyer is expected to leave store 1 earlier, seller 1 has to offer a bigger gift, and as the gift increases, the buyer’s payoff also increases. However, the buyer cannot impel seller 1 to increase this gift as much as he wants, because the buyer cannot credibly threaten seller 1 by leaving before $T^d_1$, since his initial reputation is not high enough.
However, if the buyer is the weak player in store 1, he may have to offer a probabilistic gift to seller 1 at time 0. The amount of this gift is not arbitrary. In equilibrium it must equal $z_b/X_1 e^{\lambda_1 T_1} = \tau_1$.

The gift cannot be less than this particular amount because in such a case the buyer strictly prefers accepting seller 1’s offer to finish the game at time $\tau_1$ instead of moving to the second store to play the concession game with seller 2. This contradicts the fact established in Proposition 2.1 that in equilibrium the buyer’s strategy, $F_b$, cannot have a discontinuity point in $(0, T_1)$. On the other hand, the initial gift cannot exceed this specific amount because in this case, before time $\tau_1$ the buyer’s reputation will reach the point, where it is optimal for the buyer to leave store 1. Then, seller 1 would have to adjust his strategy by making a positive probabilistic concession at time 0. That would contradict the fact that in equilibrium $F_b$ and $F_1$ cannot have a common discontinuity point in their domain (Proposition 2.1).

An indirect implication of Proposition 2.2 is that in the equilibrium strategy $\sigma^*$, if the game reaches time $T_1$, then all players assign probability 1 to the event that seller 1 is the behavioral type. As a result, the buyer leaves store 1 at time $T_1$ with probability 1 (given that the buyer’s reputation at this point is high and the traveling cost is low enough so that it is worth it to travel to the second store to negotiate with seller 2.)

If the buyer leaves store 1 at time $T_1$ with probability 1, then rational seller 1’s best response is to accept the buyer’s offer by the time $T_1$. Similarly, if seller 1’s reputation reaches 1 at time $T_1$, then leaving store 1 with probability 1 is the best response strategy for the buyer.

Now, I want to argue that in equilibrium the buyer and seller 1 cannot play a strategy that extends the concession game in store 1 beyond the time $T_1$ with some positive probability. Suppose on the contrary that each player chooses such a strategy, and these strategies establish an equilibrium. Conditional on players delaying the end of the concession game in store 1 for some extra $\hat{t}$ unit of time after $T_1$, the buyer should be indifferent between conceding to seller 1 and waiting for concession at any time $t \in [T_1, T_1 + \hat{t}]$. That is, the buyer must concede to seller 1 with a constant hazard rate during this extra time, which is a direct implication of Proposition 2.1.

However, since the buyer’s expected payoff in store 2 is a continuous and increasing
function of his own reputation, and of time, we have $1 - \alpha < \delta v_b^2(t)$ for all $t \in (T^d_1, T^d_1 + \delta]$. That is, at any time $t > T^d_1$, the buyer’s discounted continuation payoff in store 2 will be strictly higher than his instantaneous payoff in store 1. However, this contradicts the optimality of $\sigma^*$.

Therefore, conditional on each player executing a strategy that extends the concession game in store 1 after the time $T^d_1$, and these strategies constitute an equilibrium, the buyer should not concede to seller 1 with a positive (constant) hazard rate after $T^d_1$. However, this requirement implies that rational seller 1 must accept the buyer’s offer by the time $T^d_1$ with probability 1, which contradicts the initial assumption. Moreover, since the buyer’s equilibrium strategy $F^d_b$ cannot have a discontinuity point over the interval $(0, T^d_1]$ according to Proposition 2.1, the buyer cannot make a positive concession at time $T^d_1$. Thus, the event that the buyer leaves store 1 at time $T^d_1$ must occur with probability 1 in the equilibrium strategy $\sigma^*$.

As a result, I present the following Corollary with no formal proof.

**Corollary 2.1.** In the equilibrium strategy $\sigma^*$, seller 1’s reputation reaches 1 at time $T^d_1$. Moreover, if the game does not end before $T^d_1$, the buyer leaves store 1 at this time with probability 1.

Note that $\sigma^*$ is an equilibrium strategy if $\delta$ is sufficiently large (the threshold for $\delta$ is given in Appendix A). When the traveling time between the stores is large or the sellers’ demand $\alpha$ and the buyer’s demand $\alpha_b$ are very close such that it is not worth it to travel to the other store while the buyer has an option of accepting the behavioral demand of the first seller, the buyer would stay in the store he visited at time 0 and he never leaves before agreement. In such a case, players’ equilibrium strategies are: $F^d_i(t) = 1 - c_i e^{-\lambda t}$ and $F^d_b(t) = 1 - c^b_i e^{-\lambda_b t}$ with $c_i = z_i e^{\lambda T^e_i}$ and $c^b_i = z^b_i e^{\lambda^b T^e_i}$ where

$$T^e_i = \min \left\{ -\frac{\log z_i}{\lambda}, -\frac{\log z^b_i}{\lambda^b} \right\}.$$

**Proposition 2.3.** The equilibrium strategy $\sigma^*$ of the continuous-time bargaining problem $G$ is the unique (up to the buyer’s choice of store at time 0) sequential equilibrium of this game.

An important implication of Proposition 2.3 is that there is no sequential equilibrium of the game $G$ such that the buyer visits a store multiple times. Suppose on the contrary
that there is a strategy in which, without loss of generality, the buyer visits store 1 twice. Then, the buyer must be the strong player in his second visit to seller 1. Otherwise he would prefer to concede to seller 2 and finish the game before making the second visit to store 1. Thus, since seller 1 is the weak player, his payoff is $\alpha_b$ when the buyer visits his store for the second time. However, in equilibrium this continuation payoff contradicts the optimality of seller 1’s strategy because, to eliminate a further delay seller 1 would prefer to accept the buyer’s offer (for sure) when the buyer first attempts to leave his store.

The next result, which directly follows from Proposition 2.1, 2.2 and Equation (1), summarizes the weak (and rational) players’ expected payoffs in the equilibrium strategy $\sigma^*$.

**Corollary 2.2.** In the equilibrium strategy $\sigma^*$, if the buyer is the strong player in store 1 with his initial reputation, then the expected payoff to rational seller 1 is $\alpha_b$. Otherwise, the expected payoff to the rational buyer is $1 - \alpha$ (Payoffs are evaluated at time 0).

I will finish the characterization of the equilibrium strategy of the continuous-time bargaining problem G by determining the buyer’s optimal store selection at time 0. For the sake of simplicity, I assume that the sellers’ time preferences are identical (i.e. $r_1 = r_2$, or equivalently $\lambda_1 = \lambda_2 = \lambda_b$). Without loss of generality, suppose that $z_1 \geq z_2$, that is, seller 1 has a higher initial reputation. We know through Proposition 2.2 that there are two important cut-off points to evaluate the buyer’s continuation payoff in store $i \in \{1, 2\}$: $X_i z_i^{\lambda_i/\lambda}$ and $X_i$. If the buyer’s initial reputation, $z_b$, is less than the first cut-off value, then the buyer is the weak player in store $i$ (with the initial reputations). Thus the buyer’s continuation payoff evaluated at time 0, if he visits store $i$ first, is $1 - \alpha$.

However, if $z_b$ is higher than the second critical cut-off value, then the buyer is a distance-corrected strong player relative to seller $j$ (with the initial reputations) and his expected payoff evaluated at time 0 is given by

$$(1 - z_i)(1 - \alpha_b) + \delta z_i \left[ (1 - z_j e^{XT_j})(1 - \alpha_b) + z_j e^{XT_j} (1 - \alpha) \right]$$

In equilibrium, for these values of $z_b$, rational seller $i$ accepts the buyer’s offer immediately at time 0. If seller $i$ is the behavioral type, then the buyer goes directly to store $j$ and continues to play the concession game with him.
Finally, if the buyer is strong but not a distance-corrected strong player relative to seller $j$, in equilibrium the buyer is indifferent between conceding and waiting in store $i$ until the departure time $T^d_i$ and it is exactly at this time that he is indifferent between conceding to and leaving seller $i$. Moreover, according to Corollary 2.1 the buyer leaves store $i$ with probability 1 and from the time that the concession game starts in store $j$ to the time it ends, $T^e_j$, he is indifferent between conceding to seller $j$ and waiting in store $j$. As a result, the buyer’s instantaneous payoff is $1 - \alpha$ at all times. Therefore, the buyer’s expected payoff is the same in each store and is equal to what he can achieve at time 0 in store $i$. That is,

$$\left(1 - \frac{z_1 z_2}{A z^b_{\lambda/\lambda_b}}\right)(1 - \alpha_b) + \frac{z_1 z_2}{A z^b_{\lambda/\lambda_b}}(1 - \alpha)$$

Note that the buyer’s expected payoff responds to the buyer’s initial reputation $z_b$. Furthermore, the functional form of the buyer’s expected payoff differs across different intervals of $z_b$. By the definition of $X_i$ we have $X_i z^b_{\lambda/\lambda} = \left(\frac{z_1 z_2}{A}\right)^{\lambda_b/\lambda}$. Therefore, the buyer is the weak (strong) player in store 1 if and only if he is the weak (strong) player in store 2. In addition, by our assumption $z_1 \geq z_2$ we have $X_1 \leq X_2$. Therefore, if the buyer is a distance-corrected strong player relative to seller 1, he must be a distance-corrected strong player relative to seller 2, but the converse is not true.

The following result fully characterizes the equilibrium strategy of the continuous-time bargaining problem $G$.

**Proposition 2.4.** In the unique equilibrium strategy $\sigma^*$, the buyer strictly prefers to visit first the seller who has the lower reputation if the buyer is a distance-corrected strong player relative to this seller. Otherwise, the buyer is indifferent between the stores at time 0.

According to Proposition 2.2, the buyer is weak in store 1 if and only if he is weak in store 2. Thus, the weak buyer is indifferent between visiting store 1 and 2 at time 0. This is true because the buyer cannot receive a probabilistic gift from the sellers at time 0 so his payoff is $1 - \alpha$ in both stores.

When the buyer is strong but not a distance-corrected strong player relative to the seller who has a lower reputation (i.e. store 2), payoffs to the buyer in both stores are more than $1 - \alpha$, but they are equal. Thus, he is indifferent between store 1 and 2. The
initial assumption $z_1 \geq z_2$ implies that the departure time in store 1 is earlier than the departure time in store 2, i.e. $T_{d1} \leq T_{d2}$. That is, according to the equilibrium strategy the buyer is able to leave store 1 earlier than he would leave store 2. Since concession ends earlier in store 1, while at the same time seller 2 is less likely to be the behavioral type, then both sellers’ initial probabilistic gifts at time 0 happen to be the same.

If the buyer is a distance-corrected strong player relative to seller 2 but not a distance-corrected strong player relative to 1, the concession game immediately ends at time 0 in store 1 (if the buyer visits store 1 first) because the rational seller 1 immediately accepts the buyer’s offer with probability 1. However, if the buyer visits store 2 first at time 0, reaching an agreement in this store may take a significant amount of time. Nonetheless, in equilibrium the buyer picks store 2 to visit first at time 0. This is true because seller 2’s initial probabilistic gift at time 0 is strictly higher than seller 1’s initial concession in store 1.

This result seems controversial at first, especially when $z_1$ and $z_2$ are very close to each other. If the buyer visits store 2 first, he guarantees a probabilistic gift in store 1 which compensates the travel cost he has to bear when he realizes that seller 2 is the behavioral type. Therefore, the buyer’s expected payoff in each store is the same and is equal to what he can achieve in store 2 at time 0. If instead the buyer visits store 1 first and cannot reach an agreement with seller 1, the buyer does not receive a probabilistic gift in store 2 that fully compensates his traveling cost of moving from store 1 to 2. Hence, the overall expected payoff of the buyer if he visits store 1 first is less than his expected payoff if he visits store 2 first. Also, note that when $\delta$ converges to 1, the buyer’s expected payoff in each store converges to the same limit. So, being the seller who has higher initial reputation is a disadvantage especially when the buyer does not have a low reputation.

Given the values of $z_1$ and $z_2$, as $\delta$ converges to 1 the rational buyer’s payoff converges to the same limit in both stores for all values of $z_b$. If the buyer is the weak player in a store, his payoff is $1 - \alpha$ in each store for any values of $\delta$. However, if the buyer is the strong player in a store, then his expected payoff in each store converges to

$$(1 - \frac{z_1 z_2}{z_b \lambda_b})(1 - \alpha_b) + \frac{z_1 z_2}{z_b \lambda_b_b}(1 - \alpha)$$

When the buyer is the strong player, then the payoff to the rational seller that the
buyer visits at time 0 (with probability 1) is $\alpha_b$. So, the unique equilibrium outcome is inefficient. This inefficiency is due to delay in agreement and uncertainty about the types of the players. For instance, as $z_b$ also converges to 1 the rational buyer’s expected payoff converges to $(1 - \alpha_b) - z_1 z_2 (\alpha - \alpha_b)$. In this case, inefficiency is not a result of delay because for large enough $z_b$’s (relative to $z_1$ and $z_2$), the game ends at time $0^+$. Full efficiency is sustainable if $z_1$ or $z_2$ converges to 0, or $\alpha$ converges to $\alpha_b$.

Consider now the equilibrium strategies as $z_1$ increases to 1 while all other parameters remain the same. In this case, the value of $X_1$ does not change. So, if the buyer is a distance-corrected strong player relative to seller 2, he continues to be so and to choose the weak seller. However, the buyer’s expected payoff in the game decreases.

If the buyer is not a distance-corrected strong player relative to seller 2 (in the limit as $\delta$ and $z_1$ both approach 1, this is equivalent to saying that the buyer is weak in store 2) the buyer is indifferent regarding which store to go at time 0, because $T_2^d$ gets arbitrarily close to $-\frac{\log z_2}{\lambda}$, implying that $c_2$ converges to 1. That is, seller 2’s initial concession decreases to zero. Similarly seller 1’s initial concession falls to zero because the buyer becomes the weak player in store 1. Thus the payoff to the buyer decreases to $1 - \alpha$ in each store.

On the other hand, the buyer’s initial probabilistic gift to a seller increases as $z_1$ approaches 1. Therefore, in equilibrium the seller who makes the deal with the buyer gets better off as the other seller’s initial reputation increases.

When $z_2$ further decreases to 0 while all other variables remain unchanged, the buyer’s store selection in equilibrium (if it ever alters) always changes in favor of seller 2: The buyer picks the seller who has lower reputation if he is a distance-corrected strong player relative to this seller, and as $z_2$ decreases to zero the buyer becomes distance-corrected strong relative to seller 2 for all values of $z_b$ because $X_1$ decreases to zero. Therefore, having an opponent with a lower reputation is not beneficial for a seller.

According to the unique equilibrium strategy of the continuous-time bargaining problem G, we have $T_2^d \geq T_1^d$ since $z_1 \geq z_2$ by assumption. That is, the buyer would spend less time with the seller who is more likely to be the behavioral type. As a seller’s reputation tends to 1, the buyer would spend less time in his store but more time in his opponent’s store. However, in any store $i$, the buyer does not stay longer than $\tau_i$ which is the time
that seller $i$’s reputation reaches 1 if the buyer has not left, nor does seller $i$ offer any initial probabilistic gift to the buyer at time 0.

As the buyer’s initial reputation $z_b$ increases, the time that the buyer spends in a store decreases in equilibrium. For large enough values of $z_b$ (when he is a distance-corrected strong player relative to both sellers) the buyer spends almost no time in a store.

If traveling between the stores is sufficiently costly that the buyer finishes the game where he starts (or if there is just one store in the market), then the buyer would prefer to visit (or to have) a seller who has a lower reputation. Existence of a second store makes the buyer a strong player even for low values of $z_b$ and reduces the amount of time he may need to reach an agreement. Introducing a second store into the market region never makes the buyer worse off even if the new seller is stronger than the existent seller. In fact, if the buyer’s initial reputation is not too low, he becomes strictly better off.

Regarding the initial location of the buyer, for low values of $z_b$ (in particular when the buyer is not distance-corrected strong relative to the seller who has lower reputation) he strictly prefers a store that is closer to him although this store might have a higher reputation. Otherwise, there is a threshold (a function of the primitives) such that if the distance between the buyer and store 2 is above this threshold, the buyer strictly prefers store 1, i.e. the seller who is more likely to be the behavioral type.

3 Different Posted Prices

This section characterizes the equilibrium strategy of the continuous-time bargaining problem $G$ under the assumption that the sellers’ behavioral demands (posted prices) are different. Without loss of generality, I suppose that seller 1’s demand ($\alpha_1$) is strictly higher than seller 2’s demand ($\alpha_2$). I make no assumption regarding the sellers’ initial reputations unless otherwise stated.

Suppose that the buyer is in store 1 and playing the concession game with seller 1. If the buyer concedes to seller 1, the buyer’s instantaneous payoff will be $1 - \alpha_1$. However, if the buyer (immediately) leaves store 1 and goes directly to the second store to accept the behavioral demand of seller 2, his discounted (instantaneous) payoff will be $\delta(1 - \alpha_2)$. So, in equilibrium if the buyer ever visits store 1, he will never accept seller
1’s behavioral demand whenever the discount factor and the sellers’ behavioral demands satisfy \((1 - \alpha_1) \leq \delta(1 - \alpha_2)\). That is, the buyer never plays the concession game with seller 1, implying that the unique equilibrium strategy I derive in Section 2 is no longer an equilibrium strategy when the sellers’ behavioral demands are different.

However, if \((1 - \alpha_1) > \delta(1 - \alpha_2)\), then the buyer strictly prefers staying in store 1 and playing the concession game with seller 1 as long as he is the weak player in store 2. Thus, when the seller’s behavioral demands are different, the structure of the equilibrium strategy depends on the value of the discount factor, \(\delta\), relative to the difference between the sellers’ behavioral demands. The following two subsections characterize the equilibrium strategy of the continuous-time bargaining problem \(G\) under the assumption that the sellers’ behavioral demands are dissimilar.

### 3.1 Posted prices are distant relative to the distance between the stores

This subsection assumes that the discount factor is sufficiently high so that the buyer is in favor of traveling the distance between the stores to accept the behavioral demand of the second seller while he has an option of accepting the behavioral demand of seller 1. That is,

\[
\delta > \frac{1 - \alpha_1}{1 - \alpha_2} := \delta^*
\]

Notice that when the difference between \(\alpha_1\) and \(\alpha_2\) is high, \(\delta^*\) attains relatively low values. Conversely, when these two variables get arbitrarily close to one another, \(\delta^*\) approaches 1, in which case the following results hold only for the values of \(\delta\) that are very close to 1.

If \(\delta > \delta^*\) holds, then in any equilibrium of the continuous-time bargaining problem the buyer does not concede to nor spend time with seller 1 given that he ever visits store 1. Therefore, in equilibrium it must be the case that rational seller 1 instantaneously accepts the buyer’s offer with probability 1 upon his arrival, and the buyer immediately leaves store 1 if seller 1 does not concede to the buyer.

In this section, I also assume that the high-demand seller (seller 1) has a sufficiently low initial reputation. More specifically, I suppose that \(z_1 \leq \frac{\alpha_2}{\delta^*\alpha_2}\). The converse of
this inequality may hold when the buyer’s behavioral demand is too low compared to
the second seller’s behavioral demand. I defer the detailed analysis of the case where
\[ z_1 \geq \frac{\alpha_b}{\delta\alpha_2} \] to Appendix B.

Since the buyer and seller 1 plays an equilibrium strategy that impels the seller to
reveal his type immediately, the buyer’s expected payoff is \((1 - z_1)(1 - \alpha_b) + \delta z_1 v_b^2\) where
\(v_b^2\) denotes the buyer’s expected payoff in store 2 when he visits this store knowing that
seller 1 is the behavioral type. Thus, if the buyer initially chooses to visit store 2 first,
he may concede to seller 2 if and only if \(1 - \alpha_2 \geq \delta[(1 - z_1)(1 - \alpha_b) + \delta z_1 v_b^2]\) where \(1 - \alpha_2\)
is the buyer’s instantaneous payoff when he concedes to seller 2.

This relation results in a threshold
\[ z_1^* := \frac{1 - \alpha_b - \frac{1 - \alpha_2}{\delta}}{1 - \alpha_b - \delta(1 - \alpha_2)} \]
such that in equilibrium if the buyer visits store 2 first at time 0, then he plays the
concession game with seller 2 (i.e. he concedes to seller 2 with positive probability), only
if \(z_1 \geq z_1^*\). Otherwise, the buyer strictly prefers leaving store 2 immediately at time 0
(if seller 2 does not concede to him and finish the game). Therefore, in equilibrium if
\(z_1 < z_1^*\), regardless of the buyer being the strong or weak player in store 2 (according
to the initial reputations) rational seller 2 must concede to the buyer at time 0 with
probability 1.\(^{15}\) Note that \(z_1^*\) converges to 1 as \(\delta\) approaches 1. On the other hand, as \(\alpha_b\)
takes values very close to \(\alpha_2\), \(z_1^*\) may attain negative values.

I now briefly define the structure of the equilibrium strategy for the continuous-time
bargaining problem G. Define the strategy profile \(\sigma^{**}\) after a history such that the buyer
just enters store 1 at time 0, and it is the first store he is visiting. Rational seller 1
immediately accepts the buyer’s behavioral demand and finishes the game at time 0 with
probability 1. In case seller 1 is the behavioral type, i.e. seller 1 does not concede to the
buyer, the buyer immediately leaves store 1 and never comes back to this store again. The
buyer directly goes to store 2 to play the concession game with seller 2. The concession
game in store 2 continues until the time \(T_2^* = \min\{\tau_2^2, \tau_2\}\) and players concede according
to the following strategies: \(F_2(t) = 1 - c_2 e^{-\lambda_2 t}\) and \(F_2^b(t) = 1 - c_2^b e^{-\lambda_2^b t}\) where \(c_2 = z_2 e^{\lambda_2 T_2^*}\)
and \(c_2^b = z_2 e^{\lambda_2^b T_2^*}\).\(^{16}\)

\(^{15}\)The assumption \(z_1 \leq \frac{\alpha_b}{\delta\alpha_2}\) plays an important role in this inference
\(^{16}\)Note that \(\tau_2^2 = -\frac{\log z_2}{\lambda_2}\) and \(\tau_2 = -\frac{\log z_2}{\lambda_2^b}\) where \(\lambda_2 = \frac{(1 - \alpha_2) r_2}{\alpha_2 - \alpha_b}\). Also note that with some manipulation
Now define $\sigma^{**}$ after a history such that the buyer just arrives at store 2 at time 0, and it is the first store he is visiting: If seller 1’s initial reputation $z_1$ is strictly higher than $z_1^*$, then the buyer never leaves store 2. The buyer and seller 2 play the concession game in store 2 that lasts until the time $T_2^*$, and concede according to the distribution functions given in the last paragraph. If however, seller 1’s initial reputation $z_1$ is strictly less than $z_1^*$, then seller 2 immediately accepts the buyer’s behavioral demand upon his arrival. Otherwise, the buyer leaves store 2 immediately at time 0 (knowing that seller 2 is the behavioral type), and goes directly to store 1. Rational seller 1 instantly accepts the buyer’s behavioral demand with probability 1 upon the buyer’s arrival. In case seller 1 does not concede, the buyer immediately leaves his store. He directly returns to store 2, accepts the seller’s behavioral demand $\alpha_2$ with probability 1, and finalizes the game.

**Proposition 3.1.** Suppose that $\delta > \delta^*$. A strategy profile $\sigma$ of the continuous-time bargaining problem $G$ is a sequential equilibrium of this game only if $\sigma \equiv \sigma^{**}$.\(^{17}\)

I defer the proofs of all the results in this section to Appendix B.

Proposition 3.1 shows that the structure of the equilibrium strategy drastically changes (relating to the case where $\alpha_1 = \alpha_2$) when sellers’ behavioral demands are significantly different. In equilibrium, the multilateral bargaining process never ends with the buyer’s concession to the seller who has the higher demand (seller 1). If the buyer ever visits store 1, the rational seller 1 concedes to the buyer (upon the buyer’s arrival at store 1) because the buyer has the tendency to opt out instantly from the concession game in store 1.

If the buyer’s behavioral demand, $\alpha_b$, is close to the lower demand of the two sellers, $\alpha_2$, and the buyer chooses to go to this seller’s store first, then in equilibrium the buyer never leaves his store and plays the concession game with the seller until one of the players concedes.

However, if the buyer’s demand is significantly lower than $\alpha_2$ while $\delta$ is sufficiently large, and the first seller’s reputation, $z_1$, is sufficiently low, then in equilibrium the buyer immediately leaves the store he visits at time 0. As a result, rational sellers (i.e. both seller 1 and 2) finish the game by accepting the buyer’s behavioral demand at time 0.

\(^{17}\) Of the notation, I reset the clock once the buyer enters a new store.

\(^{17}\) I remark that $\sigma$ is equivalent to $\sigma^{**}$ up to the buyer’s choice of store to visit first.
Parallel to the arguments in the proof of Proposition 2.3, when the sellers’ demands are different and the discount factor $\delta$ is higher than $\delta^*$ there cannot be an equilibrium strategy in which the buyer visits the store whose seller has the higher behavioral demand, (i.e store 1) more than once. Likewise, in no equilibrium does the buyer visit the seller with the lower behavioral demand (i.e. store 2) more than twice. Along with Proposition 3.1, we can conclude that $\sigma^{**}$ is the unique (up to the buyer’s store selection at time 0) equilibrium strategy of the continuous-time bargaining problem $G$. It is unique only up to the buyer’s choice at time 0 because the buyer may be indifferent between choosing store 1 and 2 for some parameter values. Consequently, I present the following result with no formal proof.

**Corollary 3.1.** Suppose that $\delta > \delta^*$. The equilibrium strategy $\sigma^{**}$ of the continuous-time bargaining problem $G$ is the unique (up to the buyer’s store choice at time 0) sequential equilibrium of this game.

I call the buyer the locally weak player in store $i$ if $z_{ib} \leq \frac{z_i}{1 - \lambda_i}$. That is, if the buyer does not leave store $i$ until the end of the concession game in this store, local weakness of the buyer implies that seller $i$’s reputation reaches 1 faster than the buyer’s reputation, given that neither the buyer nor seller 1 makes initial probabilistic concession at time 0. Therefore, in equilibrium if the buyer is the locally weak player in store $i$ and does not leave store $i$ until the end of the concession game, the buyer must adjust his strategy by making a positive initial probabilistic concession. Thus, local weakness of the buyer in store $i$ implies his weakness in this store. However, the converse is not necessarily correct. The buyer is the locally strong player in store $i$ whenever $z_{ib} > \frac{z_i}{1 - \lambda_i}$.

**Proposition 3.2.** Suppose that $\delta > \delta^*$ and $z_1 > z_2^*$. In the unique equilibrium strategy $\sigma^{**}$, there exists some $\bar{z}_1 \in (0, 1)$ (where $\bar{z}_1 \rightarrow 1$ as $\delta \rightarrow 1$ and $\bar{z}_1 \rightarrow 0$ as $\alpha_b \rightarrow \alpha_2$) such that the buyer strictly prefers to visit store 1 (high-demand seller) first at time 0 if $z_1 \leq \bar{z}_1$. Otherwise, the buyer strictly prefers to visit store 2 first.

The value of $\bar{z}_1$ depends on whether the buyer is the locally strong or weak player in store 2. However, in either case the threshold $\bar{z}_1$ takes values arbitrarily close to 1 when the distance between the stores is sufficiently short.

In equilibrium, the buyer picks the high-demand store (store 1) to visit first if the seller’s reputation $z_1$ is high (but not too high) given that the buyer’s and the other
seller’s behavioral demands $\alpha_b$ and $\alpha_2$, respectively, are distant, or if $z_1$ is low but $\alpha_b$ and $\alpha_2$ are close (but not too close). In equilibrium, the buyer may visit store 1 for some specific values of the parameters, though he does not accept seller 1’s behavioral demand with a positive probability.

Proposition 3.2 implies that the buyer goes to the low-demand seller’s store (store 2) at time 0 if $z_1$ attains values higher than $\bar{z}_1$ because it is not worth bearing the travel cost to learn seller 1’s actual type (it is very likely that seller 1 is indeed the behavioral type). However, if $z_1$ is high but not that high (or else if it is low but the behavioral demand of seller 2 and the buyer are close, yet not too close) the equilibrium strategy of the buyer is such that the buyer learns seller 1’s actual type before he plays the concession game with seller 2.

The seemingly counterintuitive part of Proposition 3.2 (especially when $\alpha_b$ and $\alpha_2$ are distant) is that the buyer visits the high-demand seller (store 1) at time 0 even though this seller’s initial reputation is high ($z_1 > \bar{z}_1$). Note that Proposition 3.1 shows that in equilibrium, the buyer never leaves store 2 if he visits this store first given that seller 1’s reputation is high. In this case, the game may finish with some delay and the buyer never learns if seller 1 is the behavioral type or not. These factors decrease the buyer’s expected payoff in store 2. On the other hand, as $z_1$ decreases, the buyer has higher incentive to leave store 2 at any time to make the take it or leave it ultimatum to seller 1. As a result, seller 2 also has more incentive to make immediate acceptance. Thus, the payoff to the buyer if he visits store 2 first increases as seller 1’s reputation decreases. Therefore, the buyer prefers to visit store 2 at time 0 only for low values (or for very high values) of $z_1$.

Proposition 3.3. Suppose that $\delta > \delta^*$, $z_1 < z_1^*$ and the buyer is the locally strong player in the store whose seller has the lower behavioral demand (store 2). In the unique equilibrium strategy $\sigma^{**}$, there exists some $\bar{z}_b \in (0,1)$ (where $\bar{z}_b \to 1$ as $\delta \to 1$) such that the buyer strictly prefers to visit store 1 (high-demand seller) first whenever $z_b \geq \bar{z}_b$. Otherwise, the buyer strictly prefers to visit store 2 first.

Proposition 3.3 implies that if seller 1’s reputation is not too high, and the demands of the buyer and seller 2 ($\alpha_b$ and $\alpha_2$) are apart from one another, then the buyer picks store 1 to visit first only if he receives a high probabilistic gift in store 2 (after visiting
store 1). In this particular case where the primitives of the model satisfy the conditions stated in Proposition 3.3, rational seller 2 is willing to accept the buyer’s offer upon his (first) arrival at time 0, but by not visiting store 1 before making a deal in store 2, the buyer loses the chance to make the take it or leave it ultimatum to seller 1. If the probabilistic gift that the buyer will receive in store 2 after visiting store 1 is not high enough (because of low \( z_b \)), then the buyer does not risk losing the chance of receiving immediate concession by rational seller 2 at time 0 so he visits store 2 first.

As \( \delta \) approaches 1, the lower limit for \( z_b \), i.e. \( \bar{z}_b \), converges to 1. That is, when the traveling time between the stores is negligible, then the buyer goes to store 2 first if the buyer’s initial reputation is not extremely high. However, if traveling is costly, then the buyer picks store 1 at time 0 for the same values of \( z_b \) (that the buyer visits store 2 first under the higher values of \( \delta \)). By doing so he minimizes the total time he may need to spend on traveling between the stores. Note that the buyer will never accept seller 1’s behavioral demand, since \( \delta > \delta^* \). Suppose that both sellers are behavioral (which is unknown by the buyer before he visits both stores). If the buyer visits store 2 first and seller 2 does not concede to him, then according to the equilibrium strategy \( \sigma^{**} \), he must go to store 1 before conceding to seller 2. However, as traveling is costly (i.e. when \( \delta \) is low enough), the buyer does not want to travel back and forth between the stores. Thus, in equilibrium the buyer picks store 1 to visit first at time 0.

**Proposition 3.4.** Suppose that \( \delta > \delta^* \), \( z_1 < z_1^* \) and the buyer is the locally weak player in the store whose seller has the lower behavioral demand (store 2). In the unique equilibrium strategy \( \sigma^{**} \), there exists some \( \bar{z}_2 \in (0, 1) \) (where \( \bar{z}_2 \to 1 \) as \( \delta \to 1 \)) such that the buyer strictly prefers to visit store 1 (high-demand seller) first whenever \( z_2 > \bar{z}_2 \). Otherwise, the buyer strictly prefers to visit store 2 first.

According to the equilibrium strategy \( \sigma^{**} \), if the buyer visits store 2 first at time 0, then rational seller 2 accepts the buyer’s behavioral demand with probability 1 upon the buyer’s arrival at store 2. Thus, the buyer prefers to visit store 1 first at time 0 only when seller 2’s initial reputation \( z_2 \) is sufficiently high.
3.2 Posted prices are close relative to the distance between the stores

In this section, I consider the case where the discount factor $\delta$ is small relative to the difference between the sellers’ behavioral demands, i.e., $\delta \leq \delta^*$. This assumption does not necessarily imply that the discount rate is indeed very small. If the sellers’ behavioral demands are arbitrarily close to one another, we can have a high discount rate which is less than $\delta^*$.

In the previous subsection we observe that the structure of the equilibrium strategy radically changes when we relax the assumption that the seller’s demands must be equal. However, this shift in the structure of the equilibrium strategy is not “discontinuous”. This subsection highlights this point.

Under the assumption that the discount factor is less than or equal to $\delta^*$, the buyer may make deal with seller 1 in equilibrium. That is, the buyer may find it optimal to accept seller 1’s behavioral demand $\alpha_1$, with positive probability even though seller 1’s behavioral demand is higher than seller 2’s behavioral demand $\alpha_2$. This is true because if the buyer ever visits store 1, he does not leave this store unless he becomes the strong player in store 2. Therefore, in equilibrium the buyer may have to stay in store 1 and play the concession game with seller 1 to increase his reputation before he goes to store 2 to play the concession game with seller 2.

The structure of the equilibrium strategy in this section is analogous to the one that is analyzed in Section 2. Before presenting the buyer’s optimal store choice at time 0, I characterize the equilibrium strategy $\hat{\sigma}^*$ of the continuous-time bargaining problem $G$ after subgames where the buyer visits, without loss of generality, store 1 at time 0 and this is the first store he is visiting.

Let $\hat{\sigma}^* = \left( \hat{F}_1, \hat{F}_2, \hat{F}^1_b, \hat{F}^2_b, \hat{\sigma}^*_b \right)$ be a strategy profile of the continuous-time bargaining problem $G$ such that for all $i \in \{1, 2\}$ and $t \geq 0$,

\[
\hat{F}_i(t) = 1 - c_i e^{-\lambda_i t} \quad \text{and} \quad \hat{F}^i_b(t) = 1 - c^*_i e^{-\lambda^*_i t}
\]

where

\[
\lambda_i = \frac{(1 - \alpha_i) r_b}{\alpha_i - \alpha_b} \quad \text{and} \quad \lambda^*_i = \frac{\alpha_b r_i}{\alpha_i - \alpha_b}.
\]
Suppose without loss of generality that the buyer enters store 1 at time 0 and leaves store 1 at time $\hat{T}_1^d$ to go to store 2. Then
\[ c_2 = z_2e^{\lambda_2\hat{T}_2^e} \quad \text{and} \quad c_b^2 = z_b(\hat{T}_1^d)e^{\lambda_2\hat{T}_2^e} \]
where
\[ \hat{T}_2^e = \min\{-\frac{\log z_2}{\lambda_2}, -\frac{\log z_b(\hat{T}_1^d)}{\lambda_b^2}\} \quad \text{and} \quad z_b(\hat{T}_1^d) = \frac{z_b}{1 - F_b^1(\hat{T}_1^d)}. \]

**Proposition 3.5.** Suppose that $\delta \leq \delta^*$. If a strategy profile $\sigma$ of the continuous-time bargaining problem $G$ is a sequential equilibrium of $G$, then $\sigma \equiv \hat{\sigma}^*$.\(^{18}\)

**Proof.** Directly follows from Proposition 2.1. \(\square\)

**Proposition 3.6.** For $\delta$ sufficiently close to $\delta^*$, in the equilibrium strategy $\hat{\sigma}^*$ (where the buyer arrives at store 1 at time 0), the buyer is always the strong player in the concession game with the second seller. That is
\[ c_2 = z_2e^{\lambda_2\hat{T}_2^e}, \quad c_b^2 = 1, \quad \text{where} \quad \hat{T}_2^e = -\frac{\log z_b(\hat{T}_1^d)}{\lambda_b^2} \]
Moreover
\[ c_1 = z_1e^{\lambda_1\hat{T}_1^d}, \quad c_b^1 = \begin{cases} \frac{z_b}{X_1}e^{\lambda_b\hat{T}_1^d}, & \text{if } z_b < \hat{X}_1 \\ 1, & \text{otherwise}, \end{cases} \]
where the optimal time for the buyer to leave store 1 is
\[ \hat{T}_1^d = \begin{cases} \min\{-\frac{\log z_1}{\lambda_1}, -\frac{\log(z_b/X_1)}{\lambda_b}\}, & \text{if } z_b < \hat{X}_1 \\ 0, & \text{otherwise}, \end{cases} \]
such that $\hat{X}_1 = \left(\frac{z_2}{X_1}\right)^{\frac{\lambda_2}{\lambda_1}}$ and $A_1 = \frac{1 - a_b - 1 - a_1}{\alpha_2 - \alpha_b}$.

The following result is analogous to Proposition 2.3 and I present it with no formal proof.

**Corollary 3.2.** The equilibrium strategy $\hat{\sigma}^*$ of the continuous-time bargaining problem $G$ is the unique (up to the buyer’s store choice at time 0) sequential equilibrium of this game.

\(^{18}\)I remark that $\sigma$ is equivalent to $\hat{\sigma}^*$ up to the buyer’s choice of store to visit first and for each $i \in \{1, 2\}$, the parameters $c_i, c_b^i$ and $\hat{T}_i^d$ in the strategies following the subgame that the buyer arrives at store $i$ at time 0.
Since seller 1’s behavioral demand is assumed to be higher than seller 2’s, Proposition 3.5 implies that seller 2 concedes to the buyer with a higher hazard rate (relative to seller 1), i.e. \( \lambda_2 > \lambda_1 \). Moreover, if the interest rates of the sellers are the same \( (r_1 = r_2) \) we have \( \lambda_2^2 \geq \lambda_1^1 \), implying that the buyer concedes with a higher rate in store 2. To ease the notational complexity, for the remaining part of this subsection I assume that seller 1 is more patient than seller 2. In particular, suppose that \( r_1 = \frac{1 - \alpha_1}{1 - \alpha_2} r_2 \) so that we have \( \frac{\lambda_1}{\lambda_2} = \frac{\lambda_2}{\lambda_2} = \lambda_2 \).

In the unique sequential equilibrium \( \hat{\sigma}^* \), there are two threshold values for the buyer’s initial reputation \( z_b \) that determine the buyer’s optimal store election at time 0. These are \( z_b' \) and \( z_b'' \) satisfying \( z_b' < z_b < z_b'' \). For all values of \( z_b \) satisfying \( z_b' < z_b < z_b'' \) the buyer strictly prefers the high-demand store 1 at time 0.\(^{19}\) By the assumption that \( \alpha_1 \geq \alpha_2 \), \( z_b'' \) is higher than both \( \hat{X}_1 \) and \( \hat{X}_2 \). When the sellers’ behavioral demands are identical, i.e. \( \alpha_1 = \alpha_2 = \alpha \), \( z_b' \) corresponds to the threshold for \( z_b \) given in Proposition 2.2, determining whether the buyer is the strong or the weak player.

In case high-demand seller 1’s initial reputation is smaller than low-demand seller 2’s initial reputation, then the value of \( z_b'' \) is equal to 1. However, if both sellers’ demands are equal to \( \alpha \), then \( z_b'' \) corresponds to the threshold for \( z_b \) defined in Proposition 2.2 to determine if the buyer is distance-corrected strong relative to seller 2. Both \( z_b' \) and \( z_b'' \) are functions of the primitives of the model, and converge to 0 with \( z_2 \).

**Proposition 3.7.** Suppose that \( \delta \) is sufficiently close to \( \delta^* \). In the unique equilibrium strategy \( \hat{\sigma}^* \), there exists some \( z_b', z_b'' \in (0, 1) \) (where \( z_b' < z_b'' \)) such that the buyer strictly prefers to visit store 1 (high-demand seller) first whenever \( z_b' < z_b < z_b'' \). When \( z_b \) is equal to \( z_b' \) or \( z_b'' \), the buyer is indifferent between the stores. For all other values, the buyer strictly prefers to visit the low-demand seller (store 2) first.

Note that Proposition 3.7 holds under the assumption that the distance between the stores is large relative to the difference between the sellers’ behavioral demands. The next result summarizes the relation between \( \hat{\sigma}^* \) and \( \sigma^* \), the sequential equilibrium strategy profile of the continuous-time bargaining problem \( G \) under the assumption that the sellers’ demands are equal.

\(^{19}\)See Appendix B for the specific functional form of these thresholds
Proposition 3.8. As the sellers’ behavioral demands approach the same limit, the buyer’s expected payoff in the continuous-time bargaining problem \( G \) under the strategy \( \hat{\sigma}^* \) converges to his expected payoff under \( \sigma^* \). If the buyer’s store selection at time 0 is the same in both strategies, then the convergence result also holds for the sellers. Moreover, in the limit we have \( \hat{\sigma}^* = \sigma^* \).

4 The Discrete-Time Bargaining Problem

In this section, I consider the bargaining problem in discrete time and investigate the structure of its equilibria as players can make their offers increasingly frequent. I show that independent of the fine details of the bargaining protocol, equilibria of the discrete-time bargaining problem converge to the unique (up to the buyer’s store selection at time 0) equilibrium strategy of the continuous-time bargaining problem that is analyzed in Section 2.

To be more specific, I suppose that upon arrival of the buyer at store \( i \), the buyer and seller \( i \) bargain in discrete time according to a protocol \( g \) that generalizes Rubinstein’s alternating offers protocol. I denote \( g_\epsilon \) as bargaining protocol in which each player is able to make an offer within any \( \epsilon \) time interval. I show that as \( \epsilon \) converges to zero, the equilibrium outcomes of the discrete-time bargaining problem integrated with \( g_\epsilon \) converge to the unique equilibrium outcome of the continuous-time problem \( G \).

More formally, I continue to assume that there are two spatially separated stores selling an indivisible homogeneous good to a single buyer who wants to consume only one unit. In order to purchase a commodity, the buyer has to visit a store of some seller. When the buyer is in the store, he may negotiate with the seller over the surplus of size 1 according to some predetermined bargaining protocol.

In particular, a bargaining protocol \( g^i \) between the buyer and seller \( i \in \{1, 2\} \) is defined as \( g^i : [0, \infty) \to \{0, 1, 2, 3\} \) such that for any \( t \geq 0 \), an offer is made by the buyer if \( g^i(t) = 1 \) and by seller \( i \) if \( g^i(t) = 2 \). Moreover, \( g^i(t) = 3 \) implies a simultaneous offer, and \( g^i(t) = 0 \) means no offer can be made at time \( t \). An infinite horizon bargaining protocol is denoted by \( g = (g^1, g^2) \).

The bargaining protocol \( g \) is discrete. That is, for any seller \( i \in \{1, 2\} \) and for all \( \tilde{t} \geq 0 \),
the set $I^i := \{0 \leq t < \bar{t} | g^i(t) \in \{1, 2, 3\}\}$ is countable. Notice that this definition for a bargaining protocol is very general and accommodates non stationary, non alternating protocols.

Let $G(N, g)$ denote a discrete-time bargaining problem where $N = \{1, 2, b\}$ is the set of players. The discrete-time bargaining problem is defined in the following way. Initially, the buyer decides which store to visit first. Upon arrival at store $i \in \{1, 2\}$ at time 0, the buyer and seller $i$ instantaneously begin to negotiate according to the bargaining protocol $g^i$. At time $t \in I^i$, offer $x$ is made by seller $i$ if $g^i(t) = 2$, by the buyer if $g^i(t) = 1$ and by both players if $g^i(t) = 3$. An offer $x$ denotes the share the seller is to receive. If the proposer’s opponent accepts his offer, the game ends with agreement $x$ and the associated payoffs are as follows:

$$u_i(x, t, i) = xe^{-r_i}$$

for seller $i$

$$u_j(x, t, i) = 0$$

for seller $j \in \{1, 2\}$ with $j \neq i$

$$u_b(x, t, i) = (1-x)e^{-r_b}$$

for buyer

where $r_k$ is the interest rate for player $k \in N$.

If the proposer’s opponent rejects his offer, the game continues. Prior to the next offer, the buyer decides whether to stay or leave the store. If the buyer decides to stay, the next offer is made at time $t' := \min\{\hat{t} > t | \hat{t} \in I^i\}$ by the buyer if $g^i(t') = 1$. For simultaneous offers, the game $G(N, g)$ ends if the offers are compatible; in the event of strict compatibility the surplus is split equally.20

For $\epsilon > 0$ small enough, let $G(N, g_\epsilon)$ denote discrete-time bargaining problem where the buyer and the sellers bargain according to the protocol $g_\epsilon = (g_1^\epsilon, g_2^\epsilon)$ such that for all $t \geq 0$ and $i \in \{1, 2\}$, both seller $i$ and the buyer have the chance to make an offer, at least once, within the interval $[t, t+\epsilon]$ in the bargaining protocol $g_\epsilon^i$.21 In this sense, the discrete bargaining game $g_\epsilon$ converges to continuous time as $\epsilon \to 0$.22

20The buyer does not need to visit the other seller’s store to re-enter the one that he previously visited. So, for example, the buyer may change his mind while he was going to the second store and may turn back to the first one to continue negotiating with the first seller. However, the buyer will never behave that way in equilibrium.

21More formally, either $g^i(\hat{t}) = 3$ for some $\hat{t} \in [t, t+\epsilon]$, or $g^i(t') = 1$ and $g^i(t'') = 2$ for some $t, t' \in [t, t+\epsilon]$.

22I also assume that the travel time is discrete and consistent with the timing of the bargaining protocols so the buyer never arrives a store at some non-integer time.
An equilibrium outcome of the game \( G(N; g_\epsilon) \) depends on the fine details of the bargaining protocol \( g_\epsilon \) even when \( \epsilon \to 0 \). For example, let \( g'_\epsilon \) be the standard Rubinstein alternating offer bargaining protocol where \( \epsilon \) is the time that has to pass between two consecutive offers. Suppose also that whenever the buyer enters a store, he makes the first offer. Then, under the assumption that all players are perfectly rational (i.e. there are no behavioral types) and \( \epsilon \) is sufficiently close to 0, the set of subgame perfect equilibrium prices is \([0, \frac{1}{2}]\).\(^{23}\)

However, I assume that each player may be the behavioral type (as described in Section 2) with a positive probability. In this section, I assume that the behavioral demand of each seller is equal to \( \alpha \), i.e. \( \alpha_1 = \alpha_2 = \alpha \), and it is strictly bigger than \( \alpha_b \).

In the presence of these behavioral types, equilibrium outcomes of the discrete-time bargaining problem converge in distribution to the unique equilibrium outcome of its continuous-time counterpart if players can make increasingly frequent offers. This convergence result implies that the bargaining game is actually reduced to the war of attrition game, wherein each player just needs to choose whether to accept his opponent’s behavioral demand or wait for a concession. Furthermore, the players’ expected shares in equilibrium depend only on their initial reputations, time preferences and behavioral demands, not the fine details of the bargaining protocol.\(^{24}\)

The following two propositions establish this convergence result. I first show that as the discrete-time bargaining problem converges to continuous time, in any sequential equilibrium after a history that a seller is known to be rational while the buyer is not, the payoff to the buyer cannot be lower than \( \alpha_b \). Thus, in equilibrium if a seller reveals his type, he would do it by accepting the buyer’s behavioral demand. Next, I show that in any sequential equilibrium after a history where the buyer is known to be rational but sellers are not, the buyer’s expected payoff cannot be higher than \( 1 - \alpha \). So, in equilibrium if the buyer reveals his type, he does it by accepting a seller’s behavioral demand.

Therefore, by pretending to be the behavioral type, a player can force his opponent to offer his behavioral demand or to reveal his rationality. Thus, in any equilibrium, if

\(^{23}\)To make, for example, 0 an equilibrium outcome, we do not need \( \epsilon \) to be very small. As long as the travel time between the stores is less than \( \epsilon \), we can support 0 as a subgame perfect equilibrium price.

\(^{24}\)The assumption \( \alpha_1 = \alpha_2 \) is not necessary for the convergence result. It also holds if, for example, \( \alpha_1 > \alpha_2 \) so long as \( \delta \leq \delta^* = \frac{1-\alpha_1}{1-\alpha_2} \).
the buyer accepts a seller’s behavioral demand, his payoff is \(1 - \alpha\). If he waits for a seller to reveal his rationality, the buyer will get the expected payoff of \(1 - \alpha_b\). So, the highest payoff the buyer can attain is \(1 - \alpha_b\) and the lowest payoff is \(1 - \alpha\). Similarly, if a seller accepts the buyer’s behavioral demand, his payoff is \(\alpha_b\). However, if he waits for the buyer to reveal his rationality, he can attain \(\alpha\). Thus, in equilibrium the highest payoff a seller can get is \(\alpha\) and the lowest payoff (in case he makes the deal with the buyer) is \(\alpha_b\). Hence, in the limit the bargaining game turns into a concession game where each player chooses a time to concede.

**Proposition 4.1.** As \(\epsilon\) converges to zero, in any sequential equilibrium of the discrete-time bargaining problem \(G(N, g_{\epsilon})\) after any history \(h_t\) such that the buyer is in store \(i \in \{1, 2\}\) and unknown to be rational while seller \(i\) is known to be rational, the payoff to the buyer is no less than \(1 - \alpha_b - \epsilon\) and the payoff to seller \(i\) is no more than \(\alpha_b + \epsilon\) (payoffs are evaluated at time \(t\)).

I defer the proofs of the results in this section to Appendix C.

Proposition 4.1 does not claim that any \(x \geq 1 - \alpha_b\) is sustained as an equilibrium payoff for the buyer in all bargaining protocols \(g_{\epsilon}\) after the history \(h_t\) as \(\epsilon\) converges to 0. It just states that the equilibrium payoff of the buyer in a sequential equilibrium after such a history cannot be lower than \(1 - \alpha_b - \epsilon\) as players can make frequent offers.

The proof of Proposition 4.1 is very similar to the proof of Theorem 8.4 in Myerson (1991) and Lemma 1 in Abreu and Gul (2000). I basically show that as the buyer insists on mimicking the behavioral type and stays in store \(i\) long enough, in any equilibrium after the history \(h_t\), seller \(i\) gives up and accepts the buyer’s behavioral demand \(\alpha_b\) at some time \(t + t^*(\epsilon)\). Then, as players can make their offers arbitrarily frequent, seller \(i\) does not delay the game any further and accepts the buyer’s behavioral demand “almost” immediately.\(^{25}\) That is, \(t^*(\epsilon)\) converges to zero as \(\epsilon\) does.

Therefore, given that the rational buyer can secure himself the payoff close to \(1 - \alpha_b\) by not leaving the store \(i\) and mimicking the behavioral type, he does not reveal his type in an equilibrium as long as doing the opposite gives him a higher payoff.

\(^{25}\)At the earliest time possible according to the bargaining protocol \(g_i^t\).
Proposition 4.2. As $\epsilon$ converges to zero, in any sequential equilibrium of the discrete-time bargaining problem $G(N,g_{\epsilon})$ after any history $h_t$ such that the buyer is in store $i \in \{1,2\}$ and known to be rational while both sellers are not known to be rational, the payoff to the buyer is no more than $1 - \alpha + \epsilon$ and the payoff to the seller $i$ is no less than $\alpha - \epsilon$ (payoffs are evaluated at time $t$).

Proposition 4.2 claims that as $\epsilon$ converges to 0, at any sequential equilibrium of the game $G(N,g_{\epsilon})$ after the history $h_t$, the buyer makes immediate agreement with seller $i$, and the payoff to seller $i$ (which depends on the details of the bargaining protocol $g_{\epsilon}$) cannot be lower than $\alpha$ in the limit. It does not claim that every $x \geq \alpha$ is supported as an equilibrium payoff to seller $i$.

In order to prove it, I first show that if seller $i$ continues to act like the behavioral type, then the other seller will do the same with a positive probability (Lemma 4.1 and 4.2 in Appendix C). Therefore, conditional on both sellers extending the game (i.e. both sellers continue imitating the behavioral type) for long enough, the buyer will stop going back and forth between the stores at some finite time with probability 1. Along with the arguments parallel to Proposition 4.1, this result implies that the buyer does not delay the game any further, but immediately accepts seller $i$’s behavioral demand $\alpha$. As a result, in equilibrium seller $i$ does not reveal his type unless doing so gives him a payoff higher than $\alpha$.

Proposition 4.1 and 4.2 are important for seeing how a war of attrition game emerges: By pretending to be the behavioral type, the buyer can force seller $i \in \{1,2\}$ to make the offer $\alpha$ or to reveal his rationality. In either case, the buyer could obtain a payoff no less than $1 - \alpha$. This is true because by Proposition 4.1 we know that the buyer gets a payoff no less than $1 - \alpha_b$ once seller $i$ reveals himself to be rational, and $\alpha > \alpha_b$ by assumption. Thus, the buyer has a way to end the game that will yield him at least $1 - \alpha$.

On the other hand, if seller $i$ chooses to finish the game, the buyer gets a payoff at least $1 - \alpha_b$ again by Proposition 4.1. This means that in equilibrium, the buyer will get exactly $1 - \alpha_b$. But this is precisely the set up of the war of attrition game where the buyer’s low payoff is $1 - \alpha$ and his high payoff is $1 - \alpha_b$.

Similarly, by pretending to be the behavioral type, seller $i \in \{1,2\}$ forces the buyer to make the offer $\alpha_b$, to reveal his rationality or to leave seller $i$ and make a deal with the
other seller. In the latter case, seller $i$’s payoff is 0. However, in the first two cases, seller $i$ obtains a payoff no less than $\alpha_b$. This is simply by Proposition 4.2. Therefore, seller $i$ has a way to end the game that will yield him 0 or at least $\alpha_b$.

However, if the buyer chooses to finish the game, seller $i$ gets a payoff of 0 (in case the buyer makes a deal with the other seller) or at least $\alpha$, again through Proposition 4.2. This means that in equilibrium, seller $i$ will either get 0 or exactly $\alpha$. Once again, this is the set up of the war of attrition game where seller $i$’s low payoff is $\alpha_b$ and his high payoff is $\alpha$ if the buyer does not make a deal with the other seller with some positive probability.

Now, let $\sigma_\epsilon$ denote a sequential equilibrium of the discrete-time bargaining problem $G(N, g_\epsilon)$. Denote by $\sigma^0$ the buyer’s choice of store at time 0. Given $\sigma^0$, the random outcome corresponding to $\sigma_\epsilon$ is a random object $\theta_\epsilon(\sigma^0)$ which denotes any realization of an agreed division as well as a time and store at which agreement is reached.

The final result in this section shows that in the limit as $\epsilon$ converges to zero $\theta_\epsilon(\sigma^0) \rightarrow \theta(\sigma^0)$ in distribution, where $\theta(\sigma^0)$ is the unique equilibrium distribution of the game $G$ (given the buyer’s initial choice of store $\sigma^0$). Therefore, the outcome of the discrete-time bargaining problem converges in distribution to the unique (up to the buyer’s store choice at time 0) equilibrium outcome of the continuous-time bargaining problem analyzed in Section 2.

**Proposition 4.3.** As $\epsilon$ converges to 0, $\theta_\epsilon(\sigma^0)$ converges in distribution to $\theta(\sigma^0)$.

## 5 Concluding Remarks

This paper develops a reputation-based model to highlight the influence of posted prices and bargaining postures on spatially-separated duopoly competition. The introduction of behavioral types that are completely inflexible in their demands and offers, even with low probabilities, makes the equilibrium of the multilateral bargaining problem essentially unique. The equilibrium allocation does not depend on the fine details of the bargaining protocols, nor do the sellers extract all the surplus of the buyer because of the positive

\footnote{It is either store 1 or store 2.}
travel cost. Instead, it depends on the posted prices (behavioral demands) and initial reputations as well as the time preferences of the players. The equilibrium has a war of attrition structure that engenders inefficiency due to possible delay in reaching an agreement.

If the sellers’ behavioral demands are the same, then being the seller who has a higher initial reputation is a disadvantage, especially when the buyer does not have sufficiently low reputation. In equilibrium, the seller who makes the deal with the buyer gets better off as the other seller’s initial reputation increases. The buyer’s expected payoff is a decreasing function of the sellers’ initial reputations and the distance between the stores. Introducing a second store into the market region never makes the buyer worse off even if the new seller is stronger than the existing seller. In fact, if the buyer’s initial reputation is not too low, he becomes strictly better off.

When the sellers’ posted prices are different and the distance between the stores is short relative to the difference between the posted prices, the structure of the equilibrium strategy dramatically changes. In this case, the buyer never negotiates with the seller whose posted price is higher, though he may prefer to visit this store at time 0 in order to give a “take it or leave it” ultimatum. If at least one of the sellers is not a behavioral type, then in equilibrium negotiation ends almost immediately with the sellers’ concession (given that high demand seller’s initial reputation is sufficiently low). A high reputation seller would prefer to have a posted price that is higher than his opponent’s posted price, especially when the buyer has sufficiently high reputation.

However, if the distance between the stores is large relative to the difference between the posted prices, then the buyer may play the concession game with the seller that has a higher posted price, and the negotiation with this seller may take a significant amount of time. In this case, if the buyer has a high initial reputation relative to the sellers, then the buyer chooses the seller with the higher posted price to visit first at time 0. This is true because the high-demand seller offers a higher probabilistic gift to the buyer upon the buyer’s arrival at his store. Also, by visiting the high demand seller first, the buyer reduces the time that he may have to spend to reach an agreement. The buyer’s expected payoff increases with the increase in high-demand seller’s behavioral demand (as long as the inequality $\delta \leq \delta^*$ is satisfied).
Finally, in the current model, players’ posted prices and behavioral demands are exogenous, implying that the sellers are not allowed to choose their types to imitate. The model assumes perfect information regarding the players’ choices throughout the negotiation process. However, in many circumstances the sellers may not be able to attain all the information nor does the buyer convey it perfectly. Extending the model to let the players choose which of many behavioral types to imitate or introducing some informational imperfections may naturally result in different equilibrium behaviors during the negotiation process. These issues deserve comprehensive considerations and thus constitute the immediate items of my research agenda.

Appendix A

This section relaxes the restriction on strategies so the buyer can visit each store multiple times. The buyer’s strategy in game G has two parts. The first part $\sigma_b$ determines the buyer’s location at any given history. The second part is a pair of right continuous distribution functions $F_{b_1}^T$ and $F_{b_2}^{T'}$ such that for any $i \in \{1, 2\}$, history $h_T$ and interval $[T, T']$ where $T < T' \leq \infty$, we have $F_{b_i}^T : [T, T'] \rightarrow [0, 1]$.

Similarly, seller $i$’s strategy in the game G is a right continuous distribution function $F_i^T$ such that for any history $h_T$ and interval $[T, T']$ where $T < T' \leq \infty$, we have $F_i^T : [T, T'] \rightarrow [0, 1]$. Given time $t$, let $z_i(t)$ denote seller $i$’s reputation (probability that seller $i$ is the behavioral type) at time $t$. It then follows that $F_i^T(T') \leq 1 - z_i(T)$.

For any strategy profile $\sigma = \langle F; \sigma_b \rangle$ and history $h_t$, denote by $\sigma_b$ the continuation strategy after the history $h_t$. Then define $S_i^{\sigma_i}$ for each $i \in \{1, 2\}$ as the set of intervals that the buyer is in store $i$ and $I_i^{\sigma_i}$ as the collection of times that the buyer enters store $i$ according to continuation strategy $\sigma_b$. For instance, if $T \in I_i^{\sigma_i}$, then there exists a $T' > T$ such that $[T, T'] \in S_i^{\sigma_i}$, which will be interpreted as the buyer enters store $i$ at time $T$, stays there until time $T'$, and leaves store $i$ at $T'$.

Therefore, the buyer’s continuation strategy $\sigma_b$ after a history $h_t$ generates the sets $I_1^{\sigma_t}$ and $I_2^{\sigma_t}$ such that the sequence of distribution functions $\{(F_{b_1}^{T_i}, F_{b_2}^{T_i})\}_{T_i \in I_i^{\sigma_t}}$ for each $i \in \{1, 2\}$ forms the continuation strategy profile of the strategy $\sigma$ after a history $h_t$.

If $\sigma$ is an equilibrium strategy, then after any history $h_t$, $i$ and $T_i \in I_i^{\sigma_t}$ such that $[T_i, T'_i] \in S_i^{\sigma_t}$, $F_{b_i}^{T_i}$ is a best response to $F_i^{T_i}$ within the set of all right-continuous distribution functions.
\( \hat{F}_i \) defined over \([T_i, T'_i]\) such that \( \hat{F}_i(T'_i) = F_{i,T'_i}^i(T'_i) \). Same holds for \( F_i \).

For any strategy \( \sigma_b \) and history \( h_t \), let \( \tau_i = \inf \{ t \geq 0 | \exists T_i \in I^{\sigma_b}_i \ s.t \ F_{i,T_i}^i(t) = 1 - z_i(T_i) \} \) denote the time that seller \( i \)'s reputation reaches 1. Similarly, let \( \tau_b = \inf \{ t \geq 0 | F_{b,T_i}^i(t) = 1 - z_b(T_i) \} \) for some \( i \in \{1, 2\} \) and \( T_i \in I^{\sigma_b}_i \) denote the time that the buyer’s reputation reaches 1.

In equilibrium, it must be that \( \tau_i \leq \tau_b \) for any \( i \in \{1, 2\} \) because, a rational seller concedes to the buyer once he knows that the buyer is the behavioral type. If \( \tau_i = \tau_j \) where \( j \neq i \), then we have \( \tau_j = \tau_b \), because a rational buyer accepts seller \( j \)'s offer once he knows that both sellers are the behavioral type. Therefore, if \( \tau_i \leq \tau_j \), the game ends no later than time \( \tau_i \) for seller \( i \) and time \( \tau_j \) for the buyer and seller \( j \).

Suppose that \( \sigma = (F; \sigma_b) \) is a sequential equilibrium of the continuous-time bargaining problem. Pick an arbitrary history \( h_t \). Then, for any \( i \in \{1, 2\} \) and \( T_i \in I^{\sigma_b}_i \) consider the equilibrium continuation strategy profile \( (F_{i,T_i}^i, F_{T_i}^F) \). I next study the properties of these distribution functions on their domain \([T_i, T'_i] \) in \( S_i^{\sigma_b} \).

**Lemma 2.1.** Consider the equilibrium continuation strategy profile \( (F_{i,T_i}^i, F_{T_i}^F) \). If a player’s strategy is constant on some interval \([t_1, t_2]\) in the interior of its domain, then his opponent’s strategy is constant over the interval \([t_1, t_2 + \eta]\) for some \( \eta > 0 \).

**Proof of Lemma 2.1.** Here, I prove the following claim: If \( F_{i,T_i}^i(t_1) = F_{i,T_i}^i(t_2) \) for \( T_i < t_1 < t_2 < T'_i \), then \( F_{b,T_i}^i(t_1) = F_{b,T_i}^i(t_2 + \eta) \) for some \( \eta > 0 \). If \( T_i \geq \tau_i \), then the proof directly follows. So, suppose that \( T_i < \tau_i \).

Let \( F_{i,T_i}^i(t_1) = F_{i,T_i}^i(t_2) \) for \( T_i < t_1 < t_2 < T'_i \) and choose \( \epsilon \in (0, t_2 - t_1) \). For any \( t \in [t_1 + \epsilon, t_2] \), I show that the buyer does not concede in the intervening interval. Let \( U_b(t, F_{i,T_i}^i) \) and \( U_b(t_1 + \epsilon, F_{i,T_i}^i) \) denote the buyer’s expected payoff, evaluated at time \( T_i \), if he concedes to seller \( i \) at time \( t \) and \( t_1 + \epsilon \), respectively. Then, we have

\[
U_b(t, F_{i,T_i}^i) = (1 - \alpha)(1 - F_{i,T_i}^i(t))e^{-\tau,y} + \int_{T_i}^{t} e^{-\tau,y}(1 - \alpha_b)dF_{i,T_i}^i(y), \text{ and}
\]

\[
U_b(t_1 + \epsilon, F_{i,T_i}^i) = (1 - \alpha)(1 - F_{i,T_i}^i(t_1))e^{-\tau,y(t_1+\epsilon)} + \int_{T_i}^{t_1} e^{-\tau,y}(1 - \alpha_b)dF_{i,T_i}^i(y).
\]

Therefore, we have \( U_b(t, F_{i,T_i}^i) < U_b(t_1 + \epsilon, F_{i,T_i}^i) \).

Furthermore, since \( F_{i,T_i}^i \) is right-continuous, for any \( \epsilon_1 > 0 \) there is some \( \eta > 0 \) such that \( F_{i,T_i}^i(t_2 + \eta) - F_{i,T_i}^i(t_2) < \epsilon_1 \). Then, for any \( t \in (t_2, t_2 + \eta) \),

\[
U_b(t, F_{i,T_i}^i) = \int_{T_i}^{t} e^{-\tau,y}(1 - \alpha_b)dF_{i,T_i}^i(y) + (1 - \frac{\alpha + \alpha_b}{2})p_{i,T_i}^i(t) + (1 - \alpha)(1 - F_{i,T_i}^i(t))e^{-\tau,y}.
\]

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where $p^T_i(t)$ denotes the probability that seller $i$ concedes to the buyer at time $t$. Moreover,

$$
U_b(t_1 + \epsilon, F^T_i) = \int_{T_i}^{t_1+\epsilon} e^{-\tau_b y} (1 - \alpha_b) dF^T_i(y) + (1 - \alpha + \alpha_b)p^T_i(t_1 + \epsilon) + (1 - \alpha)(1 - F^T_i(t_1 + \epsilon))e^{-\tau_b(t_1+\epsilon)}
$$

(2)

Note that the second term in Equation (2) is equal to zero since $F^T_i$ is constant in this neighborhood. Therefore,

$$
U_b(t, F^T_i) - U_b(t_1 + \epsilon, F^T_i) = o(\epsilon_1) + \int_{t_1+\epsilon}^t (1 - \alpha_b)e^{-\tau_b y} dF^T_i(y)
$$

$$
+ (1 - \alpha) \left[ (1 - F^T_i(t))e^{-\tau_d t} - (1 - F^T_i(t_1 + \epsilon))e^{-\tau_b(t_1+\epsilon)} \right]
$$

$$
= o(\epsilon_1) + (1 - \alpha)\left[ e^{-\tau_d t} - e^{-\tau_b(t_1+\epsilon)} + o(\epsilon_1) \right]
$$

$$
+ \int_{t_2}^t (1 - \alpha_b)e^{-\tau_b y} dF^T_i(y)
$$

$$
= o(\epsilon_1) + o(\epsilon_1) + (1 - \alpha)\left[ e^{-\tau_d t} - e^{-\tau_b(t_1+\epsilon)} + o(\epsilon_1) \right]
$$

$$
< 0
$$

as $\epsilon_1 \to 0$.

In the first line of the equation, note that $1 - F^T_i(t_1 + \epsilon) = 1 - F^T_i(t_2)$ since $F^T_i(t_2) = F^T_i(t)$ for all $t \in [t_1, t_2]$. In addition, we can replace the lower limit of the integral by $t_2$, because $F^T_i(t_1 + \epsilon) = F^T_i(t_2)$, and hence $dF^T_i(y) = 0$ for the values of $y \in [t_1 + \epsilon, t_2]$.

Therefore, for any $t \in (t_1, t_2 + \eta]$ there is an earlier time $v \in (t_1, t)$ at which the buyer prefers to move. Hence, the buyer does not concede to seller $i$ in the interval $[t_1, t_2 + \eta]$, which implies that $F^T_{i,b}(t_1) = F^T_{i,b}(t_2 + \eta)$. Similar arguments for seller $i$ complete the proof of Lemma 2.1.

The next lemma rules out any mass point over the domain $(T_i, T'_i]$.

**Lemma 2.2.** Consider the equilibrium continuation strategy profile $(F^T_{i,b}, F^T_i)$. Then the strategy of a player does not have a mass point over $(T_i, T'_i]$.

**Proof of Lemma 2.2.** Let $p^T_i(t), p^T_{i,b}(t)$ respectively denote the probability that seller $i$ and the buyer concedes at time $t$. Here, I prove the following claim: For any $t \in (T_i, T'_i]$, we have $p^T_i(t) = p^T_{i,b}(t) = 0$.

Without loss of generality, suppose for some $t \in (T_i, T'_i]$ that $p^T_i(t) > 0$. So, for any $\epsilon > 0$ there exists some $\eta > 0$ such that for all $\tilde{t} \in [t - \eta, t)$, $F^T_i(t) - F^T_i(\tilde{t}) < p^T_i(t) + \epsilon$. Then, for $\epsilon$ and $\eta$ chosen sufficiently small and for all $\tilde{t} \in [t - \eta, t)$,

$$
U_b(t, F^T_i) = [1 - F^T_i(t)](1 - \alpha)e^{-\tau_d t} + \int_{T_i}^{\tilde{t}} (1 - \alpha_b)e^{-\tau_b y} dF^T_i(y)
$$

$$
+ (1 - \alpha + \alpha_b)p^T_i(t),
$$

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Proposition 2.1: Consider the equilibrium continuation strategy profile \((F_i^{T_i}, F_b^{T_b})\).

Proof of Proposition 2.1. Hold for seller \(i\) implying that \(U_i(\hat{t}, F_i^{T_i}) = \int_{T_i}^{\hat{t}} (1 - \alpha_t) e^{-r\theta y} dF_i^{T_i}(y)\)

implying that
\[U_b(t, F_i^{T_i}) - U_b(\hat{t}, F_i^{T_i}) > 0\]

Hence, the buyer does not concede to seller \(i\) in the interval \((t - \eta, t)\). This implies that \(F_b^{T_b}(t - \eta) = F_i^{T_i}(\hat{t})\) for all \(\eta > \hat{t} > 0\) small enough. Since we can choose \(\hat{t} > 0\) arbitrarily small, Lemma 2.1 implies that \(F_i^{T_i}(t - \eta) = F_i^{T_i}(t)\), contradicting the hypothesis \(p_i^{T_i}(t) > 0\).

Similar arguments for seller \(i\) complete the proof of Lemma 2.2.

Lemma 2.3 implies that at time \(T_i\), either seller \(i\) or the buyer (not both) can concede with positive probability.

**Lemma 2.3.** Consider the equilibrium continuation strategy profile \((F_b^{T_b}, F_i^{T_i})\). Then
\[p_i^{T_i}(T_i)p_b^{i,T_i}(T_i) = 0.\]

**Proof of Lemma 2.3.** Suppose for a contradiction that \(p_i^{T_i}(T_i)p_b^{i,T_i}(T_i) > 0\). Since \(F_i^{T_i}\) is right-continuous, for any \(\epsilon > 0\) there is an arbitrarily small \(\eta > 0\) such that \(F_i^{T_i}(T_i + \eta) - F_i^{T_i}(T_i) < \epsilon\).

Then,
\[U_b(T_i + \eta, F_i^{T_i}) = [1 - F_i^{T_i}(T_i + \eta)] (1 - \alpha_t) e^{-r\theta(T_i + \eta)} + \int_{T_i}^{T_i + \eta} (1 - \alpha_t) e^{-r\theta y} dF_i^{T_i}(y)\]
\[+ (1 - \alpha_b) e^{-r\theta t} p_i^{T_i}(T_i)\]

and
\[U_b(T_i, F_i^{T_i}) = [1 - p_i^{T_i}(T_i)] (1 - \alpha_t) + (1 - \alpha_b) p_i^{T_i}(T_i)\]

implying that
\[U_b(T_i + \eta, F_i^{T_i}) - U_b(T_i, F_i^{T_i}) > 0\]

That is, the buyer does not concede to seller \(i\) at time \(T_i\) if seller \(i\) concedes to the buyer at this time. This contradicts to the hypothesis that \(p_i^{T_i}(T_i)p_b^{i,T_i}(T_i) > 0\). Similar arguments also hold for seller \(i\).

**Proof of Proposition 2.1.** Along with Lemma 2.1-2.3, the following arguments suffice to prove Proposition 2.1: Consider the equilibrium continuation strategy profile \((F_b^{T_b}, F_i^{T_i})\). \(U_i(t, F_i^{T_i})\) denotes the expected payoff of rational seller \(i \in \{1, 2\}\) who concedes at time \(t \geq T_i\). So, if \(F_b^{T_b}\) is continuous at \(t\) then \(U_i(t, F_b^{T_b})\) is continuous at \(t\). This follows immediately from the definition of \(U_i(t, F_b^{T_b})\).
Note that there is no interval \((t', t'')\) with \(T_i \leq t' < t'' \leq T'_i \) such that both \(F^{T_i}_i\) and \(F^{i,T_i}_b\) are constant. Assume on the contrary that \(t^* < T'_i\) is the supremum of the upper bounds of \(t''\)'s such that both \(F^{T_i}_i\) and \(F^{i,T_i}_b\) are constant. However, through lemma 2.1, if \(F^{T_i}_i\) is constant on \((t', t^*)\), then \(F^{i,T_i}_b\) is constant on \((t', t^* + \eta)\) for some small but positive \(\eta\). Hence, both \(F^{i,T_i}_b\) and \(F^{T_i}_i\) are constant on this later interval, contradicting the definition of \(t^*\).

On the other hand, if \(t^* = T'_i\), then it must be that leaving store \(i\) is optimal for the buyer at time \(t^*\) (since \(\sigma_b\) is an optimal equilibrium strategy). It implies that leaving store \(i\) is also optimal for the buyer at time \(t'\), because both players' reputations are constant in this interval \([t', T'_i]\). However, in equilibrium the buyer does not bear the cost of delay and thus leaves store \(i\) at time \(t'\), contradicting \(t' < T'_i\).

Hence, if \(T_i \leq t_1 < t_2 \leq T'_i\), then we have \(F^{T_i}_i(t_2) > F^{T_i}_i(t_1)\) and \(F^{i,T_i}_b(t_2) > F^{i,T_i}_b(t_1)\). Moreover, Lemma 2.2 implies that both \(F^{T_i}_i\) and \(F^{i,T_i}_b\) are continuous over \([T_i, T'_i]\).

Then, it follows that \(D^{i,T_i}_i := \{t \mid U_i(t, F^{i,T_i}_b) = \max_{s \in [T_i, T'_i]} U_i(s, F^{i,T_i}_b)\}\) is dense in \([T_i, T'_i]\). Continuity of \(F^{i,T_i}_b\) over \([T_i, T'_i]\) implies the continuity of \(U_i(t, F^{i,T_i}_b)\) over the same domain. Hence, \(U_i(t, F^{i,T_i}_b)\) is constant for all \(t \in [T_i, T'_i]\). Consequently, \(D^{i,T_i}_i = [T_i, T'_i]\). Therefore, \(U_i(t, F^{i,T_i}_b)\) is differentiable as a function of \(t\) and we have \(\frac{dU_i(t, F^{i,T_i}_b)}{dt} = 0\). The same arguments also hold for the buyer.

For all \(t \in [T_i, T'_i]\) it must be that
\[
U_b(t, F^{i,T_i}_b) = \int_{T_i}^t (1 - \alpha_b)e^{-\tau_b x} dF^{i,T_i}_i(x) + (1 - \alpha)e^{-\tau_1 t}(1 - F^{i,T_i}_i(t))
\]
and
\[
U_i(t, F^{i,T_i}_b) = \int_{T_i}^t \alpha e^{-\tau_1 x} dF^{i,T_i}_b(x) + \alpha_b e^{-\tau_1 t}(1 - F^{i,T_i}_b(t))
\]

The differentiability of \(F^{T_i}_i\) and \(F^{i,T_i}_b\) follows from the differentiability of \(U_i(t, F^{i,T_i}_b)\) on \([T_i, T'_i]\). Differentiating them and applying Leibnitz’s rule, we get

\[
0 = (1 - \alpha_b)e^{-\tau_1 t} f^{T_i}_i(t) - (1 - \alpha)\gamma_b e^{-\tau_1 t}(1 - F^{T_i}_i(t)) - (1 - \alpha)e^{-\tau_1 t} f^{T_i}_i(t) \quad (3)
\]

and

\[
0 = \alpha e^{-\tau_1 t} f^{i,T_i}_b(t) - \alpha_b \gamma_1 e^{-\tau_1 t}(1 - F^{i,T_i}_b(t)) - \alpha_b e^{-\tau_1 t} f^{i,T_i}_b(t) \quad (4)
\]

By (3) we have

\[
e^{-\tau_1 t} f^{T_i}_i(t)[\alpha - \alpha_b] = (1 - \alpha)\gamma_b e^{-\tau_1 t}(1 - F^{T_i}_i(t))
\]

\[
\Leftrightarrow f^{T_i}_i(t) = \frac{(1 - \alpha)\gamma_b}{\alpha - \alpha_b}(1 - F^{T_i}_i(t))
\]

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Let \( \frac{(1-\alpha)b}{\alpha-\alpha_b} := \lambda \), so we have \( f_i^T_i(t) = \lambda(1 - F_i^T_i(t)) \), implying that

\[
\frac{f_i^T_i(t)}{1 - F_i^T_i(t)} = \lambda \Leftrightarrow \int_0^t \frac{f_i^T_i(t)}{1 - F_i^T_i(t)} \, dx = \int_0^t \lambda \, dx
\]

\[
\Leftrightarrow -\ln(1 - F_i^T_i(x))|_0^t = \lambda_1 t \Leftrightarrow -\ln(1 - F_i^T_i(t)) + \ln(1 - F_i^T_i(0)) = \lambda t
\]

\[
\Rightarrow F_i^T_i(t) = 1 - c_i e^{-\lambda t}
\]

where \( c_i = 1 - F_i^T_i(T_i) \).

Similarly, by 4 we have

\[
(\alpha - \alpha_b)e^{-r_i t} f_b^T_i(t) = \alpha_b r_i e^{-r_i t}(1 - F_b^T_i(t))
\]

Given that we define \( \lambda_b^i := \frac{\alpha_b r_i}{\alpha - \alpha_b} \), we have

\[
F_b^i T_i(t) = 1 - c_b^i e^{-\lambda_b^i t}
\]

where \( c_b^i = 1 - F_b^i T_i(T_i) \).

Now, consider the strategy defined in the main text where the buyer can visit each store just once. So, there is a unique \( T_1 \in I_1^b \) and \( T_2 \in I_2^b \). Moreover, given that the buyer visits store 1 first at time 0, we can find the values of \( c_2 \) and \( c_b^2 \) as follows (remark that with some manipulation of notation I also take \( T_2 = 0 \)): 

By Lemma 2.3., we know that \( F_2(0)F_2^2(0) = 0 \). First, suppose that \( F_b^2(0) = 0 \). It implies that \( c_b^2 = 1 \) and thus \( F_b^2(t) = 1 - e^{-\lambda_b^2 t} \). Therefore, \( F_b^2(T_b) = 1 - z_b(T_b) \), implying that \( 1 - e^{-\lambda_b^2 \tau_b} = 1 - z_b(T_b) \) if and only if \( \tau_b^2 = -\frac{\log z_b(T_b)}{\lambda_b^2} \).

On the other hand, if \( F_2(0) = 0 \), then we have \( c_2 = 1 \) and thus \( F_2(t) = 1 - e^{-\lambda_2 t} \). Therefore, \( F_2(T_2) = 1 - z \), implying that \( 1 - e^{-\lambda_2 \tau_2} = 1 - z \) if and only if \( \tau_2 = -\frac{\log z}{\lambda} \). Hence, the game ends at time \( T_2^* = \min\{\tau_2, \tau_b^2\} \).

**Proof of Proposition 2.2.** Suppose that players follow the strategies as dictated by \( \sigma^* \) and it is a sequential equilibrium of the continuous-time bargaining problem G. Without loss of generality suppose that the buyer first visits store \( i \) at time 0 according to \( \sigma_b^* \).

According to the equilibrium strategy \( \sigma^* \), there are three critical times that we need to take into account. For each seller \( i, j \in \{1, 2\} \) where \( i \neq j \), \( \tau_i = -\frac{\log z_i}{\lambda} \) and \( \tau_j^i = -\frac{\log z_b(T_i^d)}{\lambda_b} \). The former denotes the time that seller \( i \)'s reputation reaches 1 and the latter denotes the time that the buyer’s reputation reaches 1 in store \( j \) if he leaves store \( i \) at time \( T_i^d \).
Given that the buyer leaves store $i$ at $T^d_i$ according to the equilibrium strategy $\sigma^*_b$, the buyer’s (instantaneous) continuation payoff in store $i$ is $1 - \alpha$. However, if the buyer goes to store $j$, his continuation payoff in store $j$ will be

$$v^j_b(T^d_i) = 1 - \alpha_b - z_j e^{\lambda T^d_j} (\alpha - \alpha_b)$$

The last inequality follows from the following arguments: The buyer’s payoff in store $j$ at any time $t > 0$ is the same as his payoff at time 0, which is equivalent to

$$v^j_b(T^d_i) = F_j(0)(1 - \alpha_b) + (1 - F_j(0))(1 - \alpha)$$

$$= (1 - z_j e^{\lambda T^d_j})(1 - \alpha_b) + z_j e^{\lambda T^d_j} (1 - \alpha)$$

$$= 1 - \alpha_b - z_j e^{\lambda T^d_j} (\alpha - \alpha_b).$$

where

$$T^*_j = \min\{ - \frac{\log z_j}{\lambda}, - \frac{\log z_b(T^d_i)}{\lambda_b} \}.$$

Note that if seller $j$ is the strong player when the buyer’s reputation is $z_b(T^d_i)$, that is if $\tau_j > \tau^*_j$, then $v^j_b(T^d_i) = 1 - \alpha$. Therefore, the buyer has no incentive to leave store $i$ at time $T^d_i$ since $\delta < 1$. However, as $z_b(T^d_i)$ is an increasing function of $T^d_i$, there may exist some $T^d_i$ such that the buyer becomes the strong player in store $j$. Therefore, we need to find $T^d_i$ that yields $1 - \alpha = \delta v^j_b(T^d_i)$.

So, given that the buyer is the strong player in store $j$ if the buyer leaves store $i$ at time $T^d_i$, we have $T^c_j = - \frac{\log z_b(T^d_i)}{\lambda_b}$, which implies that

$$v^j_b(T^d_i) = 1 - \alpha_b - z_j z^\lambda_j (\alpha - \alpha_b).$$

So, $1 - \alpha = \delta v^j_b(T^d_i)$ implies that

$$A := \frac{1 - \alpha_b - \frac{1 - \alpha}{\delta}}{\alpha - \alpha_b} = z_j z_b(T^d_i) - \frac{\lambda}{\lambda_b}$$

leads to $z_b(T^d_i) = (\frac{X_i}{A})^{\lambda/\lambda_b} := X_i$. Note that for large values of $\delta$, we have $z_j^{\lambda/\lambda_b} < X_i < 1$.

Since $z_b(T^d_i) = \frac{z_b}{1 - F_b(T^d_i)} = X_i$ and $F^j_b(T^d_i) = 1 - c^j_b e^{-\lambda T^d_j}$, we have

$$c^j_b e^{-\lambda T^d_j} = \frac{z_b}{X_i} \quad (5)$$

Note that if $c^j_b$ and $T^d_i$ satisfy Equation (5), then $T^c_j = - \frac{\log z_b(T^d_i)}{\lambda_b}$. That is, the buyer is willing to leave store $i$ at time $T^d_i$. 43
There are two exhaustive cases that we need to consider about the value of $c_i$. First, consider
the case where $c_i \neq 1$, i.e. seller $i$ makes a probabilistic acceptance at $t = 0$. By Lemma 2.3
we know that, in equilibrium, the buyer does not make a probabilistic acceptance when the seller
does so. Thus, $c_i = 1$ and Equation (5) implies that

$$T_i^d = -\frac{\log(z_b/X_i)}{\lambda_i^b}$$

$T_i^d$ is well defined, i.e. $T_i^d > 0$, whenever $z_b < X_i$. Note that $T_i^d < \tau_i$ whenever $-\frac{\log(z_b/X_i)}{\lambda_i^b} < -\frac{\log z_i}{A}$ implying that $z_b > X_i z_i^{\lambda_i^b/\lambda_i}$. It is also true that $T_i^d < \tau_i^i$ because $X_i < 1$ for $\delta$ large
enough.

However, if $c_i = 1$, i.e. seller $i$ does not make an initial mass acceptance, it should be true
that $T_i^d \geq \tau_i$. In the equilibrium, the buyer does not bear the cost of delay and thus it must be
true that $T_i^d = \tau_i$. Note that for $T_i^d = \tau_i$ to be true in equilibrium it must be that $\tau_i \leq \tau_i^i$.

Moreover, since $T_i^d = -\frac{\log z_i}{X_i}$, Equation (5) implies that $c_i^d = -\frac{z_b}{X_i z_i^{\lambda_i^b/\lambda_i}}$. It is well defined
whenever (i) $z_b \leq c_i^d$ implying $X_i z_i^{\lambda_i^b/\lambda_i} \leq 1$ which is true for $\delta$ large enough, and (ii) $c_i^d \leq 1$
implying that $z_b \geq X_i z_i^{\lambda_i^b/\lambda_i}$.

Note that for $\sigma^*$ to be an equilibrium strategy, we need to have $X_i < 1$ and $A > 0$, which
holds whenever $\delta > (1 - \alpha)/(1 - \alpha_b - (\alpha - \alpha_b)z_j)$.

Therefore, for all values of $z_b$ satisfying $z_b < X_i z_i^{\lambda_i^b/\lambda_i}$, seller $i$ does not make any initial mass
acceptance, and the buyer leaves store $i$ at $T_i^d = \tau_i$. Notice that for these values of $z_b$, we have
$\tau_i \leq \tau_i^i$ as required.

Thus, for all $z_b < X_i$, the buyer leaves store $i$ at time $T_i^d = \min\{-\frac{\log z_i}{X_i}, -\frac{\log(z_b/X_i)}{\lambda_i^b}\}$. However, if $z_b \geq X_i$ then at time $t = 0$ the buyer’s discounted expected payoff in store $j$, $\delta v_j^i(0)$,
is larger than $1 - \alpha$. Therefore, for such values of $z_b$, the buyer immediately leaves store $i$ at
time 0. As a result, rational seller $i$ concedes to the buyer with probability 1 at time 0.

Since $F_i(T_i^d) = 1 - c_i e^{-\lambda T_i^d} = 1 - z_i$, we have $c_i = z_i e^{-\lambda T_i^d}$. Moreover, for $z_b \geq X_i$, we have
$c_i^d = 1$. Otherwise it must be that $c_i^d = -\frac{z_b}{X_i z_i^{\lambda_i^b/\lambda_i}}$. Therefore, it can be summarized that

$$c_i^d = \begin{cases} 
\frac{z_b}{X_i} e^{\lambda_i^b T_i^d}, & \text{if } z_b < X_i \\
1, & \text{otherwise}
\end{cases}$$

On the other hand, we have $c_j = z_j e^{\lambda T_j^d}$ and $c_i^d = 1$ where $T_j^e = -\frac{\log z_b(T_i^d)}{\lambda_i^b}$. 

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Proof of Proposition 2.3. Suppose that $\sigma^*$ is not the unique equilibrium. That is, there must exist another equilibrium strategy in which the buyer visits, for example, store 1 twice.

By Proposition 2.1, after any history $h_t$ and $T_i \in I_i^\sigma$, we know that each seller concedes to the buyer with the constant hazard rate $\lambda$ and the buyer concedes to seller $i$ with the constant hazard rate $\lambda^i_b$.

The buyer visits store 1 for the second time if and only if the buyer is the strong player in store 1. Otherwise, seller 1 does not make a probabilistic initial acceptance implying that the buyer’s expected payoff in store 1 in his second visit is $1 - \alpha$. However, since $\delta < 1$, the buyer prefers to stay in store 2 instead of visiting store 1 for the second time.

Therefore, it must be true that the buyer is the strong player when he visits store 1 for the second time. Thus, the buyer does not make initial probabilistic acceptance at his second visit, and seller 1’s expected payoff is $\alpha_b$ (evaluated through the formula $v_1 = F^i_b(T')\alpha + (1 - F^i_b(T'))\alpha_b$ where $T'$ is the time that the buyer enters store 1 for the second time and $F^i_b(T') = 0$).

However, in equilibrium seller 1 would accept the buyer’s offer when the buyer attempts to leave store 1 in his first visit. This is true because seller 1 knows that he will receive expected payoff of $\alpha_b$ when the buyer visits his store for the second time, thus he saves himself from the cost of a possible delay. Hence, there is no strategy in which the buyer visits a store multiple times.

Proof of Proposition 2.4. Consider the following four exhaustive cases regarding the value of $z_b$.

**Case I:** $z_b \leq (\frac{z_1 z_2}{A})^{\lambda_b/\lambda}$ In this case, the buyer’s expected payoff in both stores is $1 - \alpha$. Therefore, the buyer is indifferent between choosing store 1 and store 2.

**Case II:** $(\frac{z_1 z_2}{A})^{\lambda_b/\lambda} < z_b < X_1 = (\frac{z_2}{A})^{\lambda_b/\lambda}$.

In store 1 the buyer’s expected payoff is

$$U^1_b = \left[ 1 - z_1 \left( \frac{X_1}{z_b} \right)^{\lambda_b/\lambda} \right] (1 - \alpha) + z_1 \left( \frac{X_1}{z_b} \right)^{\lambda_b/\lambda} (1 - \alpha)$$

and in store 2 his expected payoff is

$$U^2_b = \left[ 1 - z_2 \left( \frac{X_2}{z_b} \right)^{\lambda_b/\lambda} \right] (1 - \alpha) + z_2 \left( \frac{X_2}{z_b} \right)^{\lambda_b/\lambda} (1 - \alpha)$$

(6)

Since $z_1 \left( \frac{X_1}{z_b} \right)^{\lambda_b/\lambda} = z_2 \left( \frac{X_2}{z_b} \right)^{\lambda_b/\lambda}$ we can conclude that $U^1_b = U^2_b$. So, the buyer is indifferent between seller 1 and seller 2.
Case III: \((\frac{z_2}{\lambda})^{\frac{\lambda_b}{\lambda}} = X_1 < z_b < X_2 = \left(\frac{z_1}{\lambda}\right)^{\frac{\lambda_b}{\lambda}}\).

In store 1, the buyer’s expected payoff is

\[
U_b^1 = \left\{ (1 - z_1) + \delta z_1 (1 - \frac{z_2}{z_b^{\lambda_b/\lambda_b}}) \right\} (1 - \alpha_b) + \frac{\delta z_1 z_2}{z_b^{\lambda_b/\lambda_b}} (1 - \alpha)
\]

and in store 2, the buyer’s expected payoff is given in Equation (6). So, \(U_b^1 \geq U_b^2\) if and only if \(z_b \leq X_1\) contradicting the assumption that \(z_b > X_1\). Hence, the buyer strictly prefers seller 2.

Case IV: \((\frac{z_1}{\lambda})^{\frac{\lambda_b}{\lambda}} = X_2 < z_b\).

Then, the buyer’s expected payoff in store 1 is given in Equation (7) while in store 2, it is

\[
U_b^2 = \left\{ (1 - z_2) + \delta z_2 (1 - \frac{z_1}{z_b^{\lambda_b/\lambda_b}}) \right\} (1 - \alpha_b) + \frac{\delta z_1 z_2}{z_b^{\lambda_b/\lambda_b}} (1 - \alpha)
\]

Since \(\delta < 1\), it must be that \(U_b^1 < U_b^2\). Therefore, the buyer picks seller 2.

Appendix B

Proof of Proposition 3.1. Consider an equilibrium strategy after a subgame where the buyer just enters store 1 at time 0 and it is the first store he visits. We know through Proposition 2.1 that each player must concede with a constant hazard rate while the buyer is in store 1. Therefore, the buyer’s instantaneous payoff of conceding to seller 1 is \(1 - \alpha_b\), which is less than \(\delta (1 - \alpha_2)\) (the lowest payoff that the buyer can get in store 2) for the assumed values of \(\delta\).

Hence, the buyer finds it optimal to go to store 2 instead of conceding to seller 1 at any given time \(t \geq 0\). Thus, in equilibrium, the buyer leaves store 1 immediately if seller 1 does not accept his offer \(\alpha_b\). So, rational seller 1 accepts the buyer’s offer with probability 1 at time 0.

Upon arrival at store 2, the buyer and seller 2 play the concession game, and Proposition 2.1 implies that the concession continues until time \(T_2^e = \min\{\frac{-\log z_2}{\lambda_2}, \frac{-\log z_2}{\lambda_2}\}\). Moreover, players concede according to the strategies \(F_2(t) = 1 - c_2 e^{-\lambda_2 t}\), \(F_2^2(t) = 1 - c_2^2 e^{-\lambda_2^2 t}\) where \(c_2 = z_2 e^{\lambda_2 T_2^e}\) and \(c_2^2 = z_2 e^{\lambda_2^2 T_2^e}\).

It then implies that the buyer’s expected payoff (evaluated at time 0) in this subgame is \(U_b^1 = (1 - z_1)(1 - \alpha_b) + \delta z_1 u_b^2\) where \(u_b^2 = (1 - F_2(0))(1 - \alpha_2) + F_2(0)(1 - \alpha_b)\). In particular, if the buyer is the locally-strong player in store 2, i.e., \(z_b \geq z_2^{\lambda_2/\lambda_2}\) then \(u_b^2 = z_2 z_2^{-\lambda_2/\lambda_2^2} (1 - \alpha_2) + (1 - z_2 z_2^{-\lambda_2/\lambda_2^2})(1 - \alpha_b)\). Otherwise, for all \(z_b < z_2^{\lambda_2/\lambda_2}\) we have \(u_b^2 = 1 - \alpha_2\).

Now consider an equilibrium strategy after a subgame that the buyer just enters store 2 at time 0, and it is the first store he is visiting. Again, by Proposition 2.1 we know that each
player must concede with a constant hazard rate while the buyer is in store 2. If the buyer concedes to seller 2, his instantaneous payoff is \(1 - \alpha_2\). However, if the buyer leaves store 2 at some time \(t \geq 0\) and goes to store 1, we know from previous arguments that concession in store 1 will immediately finish upon arrival of the buyer. So, the buyer would directly come back to store 2 if seller 1 is the behavioral type. Thus, the buyer’s continuation payoff if he leaves store 2 at time \(t\) is \((1 - z_1)(1 - \alpha_b) + \delta z_1 v_b^2\) where \(v_b^2 = (1 - F_2(0))(1 - \alpha_2) + F_2(0)(1 - \alpha_b)\) denotes the buyer’s expected payoff in his second visit to store 2.\(^{27}\)

\(v_b^2 > 1 - \alpha_2\) requires that seller 2 offers positive probabilistic gift to the buyer on his second visit. In this case, seller 2’s expected payoff must be \(\alpha_b\). However, optimality of the equilibrium strategy implies that rational seller 2 should accept the buyer’s offer with probability 1 when the buyer attempts to leave his store for the first time. Hence, it must be that in equilibrium \(v_b^2 = 1 - \alpha_2\). Therefore, the buyer’s payoff if he leaves store 2 is

\[
v_b^1 = \delta [(1 - z_1)(1 - \alpha_b) + \delta z_1(1 - \alpha_2)].
\]

If \(v_b^1\) is larger than \(1 - \alpha_2\), then the buyer leaves store 2 immediately at time 0 instead of conceding to seller 2. However, \(v_b^1 > 1 - \alpha_2\) implies that \(z_1 < \frac{1 - \alpha_0 - \frac{1 - \alpha_2}{1 - \alpha_b}}{1 - \alpha_b - \delta(1 - \alpha_2)} := z_1^*\).

Note that \(z_1^* < 0\) whenever \(\delta \leq \frac{1 - \alpha_2}{1 - \alpha_b}\). In this case, the buyer never leaves store 2. So, the concession game between the buyer and seller 2 finishes at time \(T_2^e = \min\{\tau_2^2, \tau_2\}\). If \(\delta > \frac{1 - \alpha_2}{1 - \alpha_b}\) holds, then \(z_1^* \in (0, 1)\) and it converges to 1 from below as \(\delta \to 1\). Therefore, for all values of \(z_1\) where \(z_1 > z_1^*\), the buyer finds it optimal to stay in store 2 and continue playing the concession game with seller 2 until time \(T_2^e = \min\{\tau_2^2, \tau_2\}\). Otherwise, for all values of \(z_1 < z_1^*\), the buyer finds it optimal to leave store 2 immediately at time 0.

Next, I determine seller 2’s optimal strategy when the buyer intends to leave store 2 upon his arrival (i.e. \(z_1 < z_1^*\)).

If the buyer is the strong player in store 2, i.e. \(\tau_2^2 \leq \tau_2\), then optimality of the equilibrium strategy implies that rational seller 2 must accept the buyer’s offer at time 0 with probability 1. In this particular case, note that the buyer’s payoff is

\[
(1 - z_2)(1 - \alpha_b) + z_2\delta[(1 - z_1)(1 - \alpha_b) + z_1\delta(1 - \alpha_2)]
\]

Consider now the case where seller 2 is the locally-strong player in store 2 (i.e., \(\tau_2^2 > \tau_2\)). Consider seller 2’s payoff when the buyer arrives store 2 for the second time. I denote it by \(\hat{V}_2\).\(^{27}\) Corollary 3.1 shows that there is no equilibrium in which the buyer visits store 1 multiple times and store 2 more than twice.
It is an increasing function of seller 2’s reputation at the time that the buyer enters store 2 for
the second time. Note that $\hat{V}_2$ is not more than $\alpha_2$.

If seller 2 accepts the buyer’s offer at time 0, his payoff will be $\alpha_b$. If he does not concede
to the buyer but allows him to leave store 2 at his first visit, seller 2’s expected payoff will be $\delta^2 z_1 \hat{V}_2$, which is no more than $\delta^2 z_1 \alpha_2$. However, by assumption we have $\alpha_b \geq \delta^2 z_1 \alpha_2$, which implies that rational seller 2 immediately concedes to the buyer with probability 1 upon the
buyer’s arrival at store 2 at time 0. Therefore, note that the buyer’s payoff by visiting store 2
first is

$$(1 - z_2)(1 - \alpha_b) + z_2 \delta[(1 - z_1)(1 - \alpha_b) + z_1 \delta(1 - \alpha_2)].$$

Proof of Proposition 3.2. Suppose that $z_1 > z_1^*$. If the buyer first visits store 1, then his
expected payoff is $U_b^1 = (1 - z_1)(1 - \alpha_b) + \delta z_1 u_b$. However, if he chooses to visit store 2 first,
his expected payoff is $u_b$.

Whenever the buyer is the strong player in store 2, $u_b$ is equal to $z_2 z_b^{-\lambda/\lambda_b}(1 - \alpha_2) + (1 -
\delta z_2 z_b^{-\lambda/\lambda_b})(1 - \alpha_b)$. Otherwise it is equal to 1 $- \alpha_2$. So, $U_b^1 > u_b$ implies that

$$z_1 \leq \bar{z}_1 := \frac{1 - \alpha_b - u_b}{1 - \alpha_b - \delta u_b}.$$

Hence the buyer selects store 1 to visit first whenever $z_1 \leq \bar{z}_1$.

Proof of Proposition 3.3. Suppose that $z_1 < z_1^*$ and the buyer is the strong player in store
2, that is $\tau_b^2 < \tau_2$ equivalently, $z_b > z_2^{\lambda_2/\lambda_2}$. If the buyer first visits store 1, then his expected
payoff is $U_b^1 = (1 - z_1)(1 - \alpha_b) + \delta z_1 u_b$ where $u_b = z_2 z_b^{-\lambda_2/\lambda_b}(1 - \alpha_2) + (1 -
\delta z_2 z_b^{-\lambda_2/\lambda_b})(1 - \alpha_b)$. However, if he chooses to visit store 2 first, his expected payoff is $U_b^2 = (1 - z_2)(1 - \alpha_b) + z_2 \delta[(1 -
\delta z_1 z_2(1 - \alpha_b)(\delta - z_b^{-\lambda_2/\lambda_2})$$ (8)

which is equivalent to

$$z_b \geq \bar{z}_b := \left(\frac{\delta z_1 z_2(\alpha_2 - \alpha_b)}{|z_2 - z_1|(1 - \delta)(1 - \alpha_b) + \delta z_1 z_2(1 - \alpha_b - \delta(1 - \alpha_2))}\right)^{\lambda_2/\lambda_2}$$

(9)

Note that the right hand side of this inequality is always less than 1 when $z_2 \geq z_1$. When
$z_1 > z_2$, it is less than 1 whenever $1 \geq z_2 > \frac{z_1(1 - \alpha_b)}{1 - \alpha_b + z_1(1 - \alpha_2)}$.

Notice that as $\delta$ converges to 1, $\bar{z}_b$ converges to 1. That is, in the limit of $\delta$, the buyer does
not prefer store 1. But, for relatively smaller $\delta$’s the buyer goes to store 1 if $z_b$ is large enough.
Proof of Proposition 3.4. Suppose now that seller 2 is the locally-strong player, that is \( \tau^2_b \geq \tau_2 \), equivalently, \( z_b \leq z_2^b/\lambda_2 \). If the buyer first visits store 1, then \( U^1_b = (1 - z_1)(1 - \alpha_b) + \delta z_1(1 - \alpha_2) \). However, if the buyer first visits store 2, then the buyer’s expected payoff in store 2 is \( U^2_b = (1 - z_2)(1 - \alpha_b) + z_2 \delta[(1 - z_1)(1 - \alpha_b) + z_1 \delta(1 - \alpha_2)] \). So, the buyer picks store 1 to go first if, and only if \( U^1_b \geq U^2_b \) which holds whenever

\[
 z_2 \geq z^*_2 := \frac{1 - \alpha_b - (1 - z_1)(1 - \alpha_b) - \delta z_1(1 - \alpha_2)}{1 - \alpha_b - \delta[(1 - z_1)(1 - \alpha_b) + \delta z_1(1 - \alpha_2)]}.
\]

Note that the right hand side of this inequality is always less than 1 and converges to 1 as \( \delta \to 1 \). Hence, in the limit, the buyer will never select store 1, but for smaller values of \( \delta \) and large values of \( z_2 \), the buyer chooses store 1 to go first.

The Case Where \( z_1 \geq \frac{\alpha_b}{\delta^* \alpha_2} \)

Suppose now that seller 1’s reputation is high enough (or the buyer’s behavioral demand is low enough) so that \( \delta^2 z_1 \alpha_2 > \alpha_b \) holds. The structure of the equilibrium strategy \( \sigma^{**} \) does not change if the buyer is locally strong in store 2. In this case, in equilibrium seller 2 accepts the buyer’s behavioral demand immediately upon the buyer’s arrival in store 2. However, \( \sigma^{**} \) might slightly change when seller 2 is the strong player in store 2. The small modification we need to make is the following. If the buyer visits seller 2 first and if \( z_1 < z^*_1 \), then seller 2 accepts the buyer’s behavioral demand upon the buyer’s arrival at store 2 with some probability \( 1 - \mu \) for some \( \mu \in [z_2, 1] \). That is, \( F^{(1)}_2(0) = 1 - \mu \) where \( F^{(1)}_2(\cdot) \) denotes seller 2’s strategy when the buyer visits store 1 for the first time. However, in case seller 2 does not accept the buyer’s behavioral demand at time 0, seller 2 and the buyer play the concession game (in case the buyer returns back to store 2) until the time \( T^*_2 = \min\{z^*_2, \tau^*_2\} \) with the distribution functions \( F^{(2)}_b(t) = 1 - c^*_b e^{-\lambda^*_2 t} \) and \( F_2(t) = 1 - c^*_2 e^{-\lambda_2 t} \) where \( c^*_2 = z^*_2 e^{\lambda_2 T^*_2} \) and \( c^*_b = z^*_b e^{\lambda^*_2 T^*_2} \). The rest of the new (modified) strategy follows directly from \( \sigma^{**} \).

Next, I will argue that in equilibrium \( \mu \) can take only two values.

Case 1. Suppose that \( \mu = z^*_2 \). Then in equilibrium seller 2’s payoff is \( \alpha_b \). However, if rational seller 2 deviates and does not accept the buyer’s demand at time 0 but waits for the buyer’s second visit to his store, his expected payoff would be \( z_1 \delta^2 \alpha_2 \) (since in the second visit, the buyer would assign probability 1 that seller 2 is the behavioral type.) Since we have \( \alpha_b < z^*_1 \delta^2 \alpha_2 \) by assumption, seller 2 would deviate from his equilibrium strategy. Hence, for \( \mu = z^*_2 \), the modified strategy of \( \sigma^{**} \) cannot establish an equilibrium.
**Case 2.** Suppose that $\mu = 1$. That is, seller 2 does not concede to the buyer with a positive probability at time 0. In this case, seller 2’s expected payoff is $z_1 \delta^2 V_2$ where $V_2$ is seller 2’s expected payoff when the buyer enters store 2 for the second (and the last) time. Since $\mu = 1$, seller’s reputation at the buyer’s second visit is still $z_2$. Therefore given that $V_2 = F_b^{2,2}(0)\alpha_2 + (1 - F_b^{2,2}(0))\alpha_b$ where $F_b^{2,2}(0) = 1 - z_b e^{\lambda_2^2 T_2^e}$ with $T_2^e = \frac{-\log z_2}{\lambda_2}$ such that $F_b^{2,2}(0)$ denotes the buyer’s initial concession when he enters store 2 for the second time. Therefore, $V_2 = (1 - z_b z_2^{-\lambda_2^2/\lambda_2})\alpha_2 + z_b z_2^{-\lambda_2^2/\lambda_2} \alpha_b$.

Thus, seller 2 does not concede to the buyer at time 0 if and only if $\delta^2 z_1 V_2 > \alpha_b$, which is equivalent to

$$z_b \leq z_b^* := \frac{\delta^2 z_1 \alpha_2 - \alpha_b}{\alpha_2 - \alpha_b} \delta^2 z_1 z_2^{-\lambda_2^2/\lambda_2}.$$

Hence, in equilibrium seller 2 does not concede to the buyer at time 0, but waits his second visit to play the concession game until time $T_2^e$ as long as $z_b \leq z_b^*$.

**Case 3.** Now suppose that $\mu \in (z_2, 1)$. In this case, seller 2’s expected payoff of not conceding to the buyer at time 0 is $z_1 \delta^2 V_2(\mu)$ where $V_2(\mu)$ is seller 2’s expected payoff when the buyer enters store 2 for the second time. Seller 2’s reputation when the buyer enters store 2 for the second time will be $z_2/\mu$.

Seller 2’s payoff of conceding at time 0 is $\alpha_b$. His payoff of not conceding but waiting to play the concession game with the buyer with a higher reputation is $z_1 \delta^2 V_2(\mu)$ where

$$V_2(\mu) = (1 - z_b \frac{z_2}{\mu} z_2^{-\lambda_2^2/\lambda_2})\alpha_2 + z_b \frac{z_2}{\mu} z_2^{-\lambda_2^2/\lambda_2} \alpha_b.$$

Since, seller 2 randomizes between conceding at time 0 and waiting the buyer’s second appearance, seller 2’s expected payoff in these two cases must be the same. That is, $\delta^2 z_1 V_2(\mu) = \alpha_b$, implying that

$$\mu = \left( \frac{\delta^2 \alpha_2 z_1 - \alpha_b}{\delta^2 z_1 z_2 (\alpha_2 - \alpha_b)} \right) \frac{\lambda_2^2}{z_b} z_2 = \left( \frac{z_b^*}{z_b} \right) \frac{\lambda_2^2}{\lambda_2^2}.$$

Hence, in the other equilibrium, seller 2 is indifferent between accepting the buyer’s behavioral demand at time 0 and waiting for his second visit. As a result, seller 2 concedes to the buyer with probability $1 - \mu$ at time 0. If $z_b \leq z_b^*$, then $\mu \geq 1$, which contradicts the assumption about the value of $\mu$. Thus, this modified strategy is establishes an equilibrium for the values of $z_b$ which are strictly higher than $z_b^*$. 

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The Characterization of the equilibrium strategies when \( z_1 \geq \frac{\alpha_b}{\delta \alpha_2} \)

Note that if the buyer is locally strong in store 2, the equilibrium strategy \( \sigma^{**} \) does not change. Hence, Proposition 3.3. suffices in this case.

If instead the buyer is locally weak in store 2, then the buyer picks store 1 to visit first at time 0 given that seller 2 does not concede to the buyer when the buyer first visits store 2 and \( z_b \leq z_b^* \) holds.

If the buyer is locally weak in store 2 and seller 2 concedes to the buyer when he first arrives store 2 with probability \( 1 - \mu \), then the buyer selects store 1 at time 0 whenever the following holds:

\[
z_b^* < z_b \leq z_b^* \left[ 1 - \alpha_b - \frac{\delta[(1 - z_1)(1 - \alpha_b) + \delta z_1(1 - \alpha_2)]}{z_1[1 - \alpha_b - \delta(1 - \alpha_2)]} \right]^{\lambda_j/\lambda_j} \]

given that the right hand side of this inequality is strictly higher than \( z_b^* \).

**Proof of Proposition 3.6.** Suppose that players play according to \( \hat{\sigma}^* \) and it is a sequential equilibrium of the continuous-time bargaining problem G. Without loss of generality, suppose that the buyer first enters store \( i \) at time 0.

The buyer’s instantaneous payoff in store \( i \) is \( 1 - \alpha_i \). If he goes to store \( j \), his continuation payoff will be \( v^j_b(\hat{T}_i^d) = 1 - \alpha_b - z_je^{\lambda_j}(\alpha_j - \alpha_b) \) where \( \hat{T}_j = \min\{ -\log z_j/\lambda_j, -\log z_b(\hat{T}_i^d)/\lambda_b \} \).

If seller \( j \) is the strong player in store \( j \) while the buyer’s reputation is \( z_b(\hat{T}_i^d) \), then \( v^j_b(\hat{T}_i^d) = 1 - \alpha_j \) which is less than \( \delta(1 - \alpha_i) \) since \( \delta \) is less than \( (1 - \alpha_1)/(1 - \alpha_2) \), or if \( j = 1, \alpha_1 \) is more than \( \alpha_2 \). Therefore, the buyer must be the strong player in store \( j \) if he leaves store \( i \) at time \( \hat{T}_i^d \). This implies that \( v^j_b(\hat{T}_i^d) = 1 - \alpha_b - z_jz_b(\hat{T}_i^d)^{\lambda_j/\lambda_j}(\alpha_j - \alpha_b) \).

Moreover, if the buyer leaves store \( i \) at time \( \hat{T}_i^d \) then it must be that \( 1 - \alpha_i = \delta v^j_b(\hat{T}_i^d) \). The last equality implies that

\[
z_jz_b(\hat{T}_i^d)^{-\lambda_j/\lambda_b} = \frac{1 - \alpha_b - \frac{1 - \alpha_1}{\delta \alpha_2}}{\alpha_j - \alpha_b} := A_i
\]

which leads to \( z_b(\hat{T}_i^d) = \left( \frac{z_j}{X_i} \right)^{\lambda_j/\lambda_j} := \hat{X}_i \) and since \( \hat{T}_i^j(\hat{T}_i^d) = 1 - c_i^j e^{-\lambda_i \hat{T}_i^d} \), we have

\[
c_i^j e^{-\lambda_i \hat{T}_i^d} = \frac{z_b}{X_i}
\]

Note that for large enough \( \delta \) as long as \( \delta \leq \frac{1 - \alpha_1}{1 - \alpha_2} \), we have \( z_j^{\lambda_j/\lambda_j} < \hat{X}_i < 1 \). The rest of the arguments follow from the proof of Proposition 2.2.
Proof of Proposition 3.7. First, parallel arguments to the proof of Proposition 2.3 imply that if $\hat{\sigma}^*$ is a sequential equilibrium, it is the only equilibrium (up to the buyer’s store selection at time 0) of the game $G$.

Second, the buyer’s expected payoff calculation at time 0 varies with the value of $z_b$. For any seller $i \in \{1, 2\}$, if $z_b \leq \hat{X}_i z_i^{\lambda_b/\lambda}$, denote it region $I$, the buyer’s expected payoff of visiting store $i$ at time 0 is $U^i_b(I) = 1 - \alpha_i$. If $\hat{X}_i z_i^{\lambda_b/\lambda} < z_b < \hat{X}_i$, denote it as region $II$, then the buyer’s expected payoff $U^i_b(II)$ is equal to

$$U^i_b(II) = \left(1 - \frac{z_1 z_2}{A_1 z_b^{\lambda/\lambda_b}}\right)(1 - \alpha_b) + \frac{z_1 z_2}{A_1 z_b^{\lambda/\lambda_b}} (1 - \alpha_i).$$

Finally, if $z_b \geq \hat{X}_i$, region $III$, his payoff is equal to

$$U^i_b(III) = \left[(1 - z_i) + \delta z_i (1 - \frac{z_j}{z_b^{\lambda/\lambda_b}})\right](1 - \alpha_b) + \frac{\delta z_2}{z_b^{\lambda/\lambda_b}} (1 - \alpha_j).$$

First suppose that $z_2 \geq z_1 \frac{A_1}{A_2}$. Along with the assumption $\alpha_1 \geq \alpha_2$, we have $\hat{X}_1 > \hat{X}_2$ and $\hat{X}_1 z_1^{\lambda_b/\lambda} < \hat{X}_2 z_2^{\lambda_b/\lambda}$. Therefore, there are 4 cut-off values (and 5 regions) that we need to consider. These are: $\hat{X}_1 z_1^{\lambda_b/\lambda}$, $\hat{X}_2 z_2^{\lambda_b/\lambda}$, $\hat{X}_2$ and $\hat{X}_1$.

**Case 1.** $z_b \leq \hat{X}_1 z_1^{\lambda_b/\lambda}$. In store 1 the buyer receives the expected payoff of $1 - \alpha_1$ and in store 2, $1 - \alpha_2$. Hence, the buyer chooses store 2 at time 0.

**Case 2.** $\hat{X}_1 z_1^{\lambda_b/\lambda} < z_b \leq \hat{X}_2 z_2^{\lambda_b/\lambda}$. Then the buyer’s expected payoff in store 1 is larger than his payoff in store 2, i.e. $U^1_b(II) > 1 - \alpha_2$ if and only if

$$z_b > \left(\frac{z_1 z_2 (\alpha_1 - \alpha_b)}{A_1 (\alpha_2 - \alpha_b)}\right)^{\lambda_b/\lambda} := z'_b$$

which is less than $\hat{X}_2 z_2^{\lambda_b/\lambda}$.

**Case 3.** $\hat{X}_2 z_2^{\lambda_b/\lambda} \leq z_b < \hat{X}_2$. In this case we have

$$U^1_b(II) = \left(1 - \frac{z_1 z_2}{A_1 z_b^{\lambda/\lambda_b}}\right)(1 - \alpha_b) + \frac{z_1 z_2}{A_1 z_b^{\lambda/\lambda_b}} (1 - \alpha_1)$$

$$U^2_b(II) = \left(1 - \frac{z_1 z_2}{A_2 z_b^{\lambda/\lambda_b}}\right)(1 - \alpha_b) + \frac{z_1 z_2}{A_2 z_b^{\lambda/\lambda_b}} (1 - \alpha_2)$$

So, $U^1_b(II) > U^2_b(II)$ whenever $(1 - \alpha_b)(\frac{1}{A_2} - \frac{1}{A_1}) > \frac{1 - \alpha_2}{A_2} - \frac{1 - \alpha_1}{A_1}$ which yields that $A_1 (\alpha_2 - \alpha_b) > A_2 (\alpha_1 - \alpha_b)$. The last inequality is true if and only if $\alpha_1 > \alpha_2$. Therefore, for all values of $z_b$, the buyer selects store 1 at time 0.
Case 4. \( \hat{X}_2 \leq z_b < \hat{X}_1 \). Then we have

\[
U^1_b(II) = \left(1 - \frac{z_1z_2}{A_1z_b^{\lambda/\lambda_b}}\right)(1 - \alpha_b) + \frac{z_1z_2}{A_1z_b^{\lambda/\lambda_b}}(1 - \alpha_1)
\]

\[
U^2_b(III) = \left[\left(1 - z_2\right) + \delta z_2(1 - \frac{z_1}{z_b^{\lambda/\lambda_b}})\right](1 - \alpha_b) + \frac{\delta z_1z_2}{z_b^{\lambda/\lambda_b}}(1 - \alpha_1)
\]

We have \( U^1_b(II) > U^2_b(III) \) whenever

\[
z_b > \left(\frac{\lambda - \delta}{1 - \lambda}\right) (1 - \alpha_1) z_1 \lambda/\lambda_b \left[\frac{1}{1 - \delta} \right]
\]

If we can show that the right hand side of this inequality is less than \( \left(\frac{z_1}{A_1}\right)^{\lambda_b/\lambda} \), which is less than \( \left(\frac{z_2}{A_2}\right)^{\lambda_b/\lambda} = \hat{X}_2 \), we reach to the desired conclusion since \( z_b \geq \hat{X}_2 \).

So, suppose for a contradiction that \( \frac{\lambda - \delta}{1 - \lambda} (1 - \alpha_1 - \delta z_1 z_2) \geq \hat{X}_2 \). It then yields that \( \frac{1}{A_1}(1 - \alpha_2 - \lambda z_1 z_2) \geq \delta (1 - \alpha_1 - \alpha_2) \) which is equivalent to \( \frac{1}{A_1} \delta (1 - \alpha_1 - \frac{\epsilon}{\delta} A_1 \lambda) \). Hence, the buyer prefers store 1 at time 0 for all values of \( z_b \).

Case 5. \( \hat{X}_1 \leq z_b \). In this case \( U^1_b(III) > U^2_b(III) \) requires that \( \frac{\delta z_1z_2}{z_b^{\lambda/\lambda}} (1 - \alpha_2) > (1 - \alpha_b)(z_1 - z_2)(1 - \delta) \). So, as \( z_2 \geq 1\frac{A_2}{z_b} \) this inequality holds for all values of \( z_b \), implying that the buyer prefers store 1 at time 0 for all values of \( z_b \).

Now, suppose that \( z_2 < 1\frac{A_2}{z_b} \). So we have \( \hat{X}_1 z_1^{\lambda_b/\lambda} < \hat{X}_2 z_2^{\lambda_b/\lambda} \), \( \hat{X}_1 < \hat{X}_2 \) and \( \hat{X}_2 z_2^{\lambda_b/\lambda} < \hat{X}_1 \) since \( A_1 < 1 \) and for assumed values of \( \delta, \hat{X}_2 < 1 \). When the previous arguments are applied to this case, we reach the same conclusions. However, assumption \( z_2 < 1\frac{A_2}{z_b} \) covers the cases where \( z_2 < z_1 \). So, for the values of \( z_b > \hat{X}_2 \), the buyer chooses store 1 only when \( z_b \leq \left( \frac{\delta z_1z_2(1 - \alpha_2)}{(1 - \alpha_b)(z_1 - z_2)(1 - \delta)} \right)^{\lambda_b/\lambda} \). Hence, define \( z''_b \) as follows:

\[
z''_b = \begin{cases} 
1 & \text{if } 0 < z_1 \leq z_2 \\
\max\{\hat{X}_1, \left( \frac{\delta z_1z_2(1 - \alpha_2)}{(1 - \alpha_b)(z_1 - z_2)(1 - \delta)} \right)^{\lambda_b/\lambda} \} & \text{otherwise}
\end{cases}
\]

This finishes the proof of Proposition 3.7.

Proof of Proposition 3.8. The proof of the claim for the buyer follows directly from the proof of Proposition 3.7. Note that as \( \alpha_i \)’s converge to \( \alpha \), the buyer’s payoff in each store converges to the same limit as long as \( z_b \leq z_2 \), given that \( z_2 \geq 1\frac{A_2}{z_b} \). Moreover, in the limit \( A_i \)’s converge to \( A \) and \( \hat{X}_2 \) converges to \( X_2 \).

For the values of \( z_b \) satisfying \( z_b > \hat{X}_2 \), the buyer continues to select the seller who has the lower reputation (seller 1 in this case). Same arguments for \( z_2 < 1\frac{A_2}{z_b} \) prove the convergence for the buyer. Similar arguments for the sellers complete the proof.
Appendix C

**Proof of Proposition 4.1.** This proof is adapted from the proof of Theorem 8.4 in Myerson (1991) and Lemma 1 in Abreu and Gul (2000).

I show that the payoff to the buyer if he continues to stay in store \(i\) and mimics the behavioral type converges to \(1 - \alpha_b\) as \(\epsilon\) converges to zero. Given this, we can conclude that in any sequential equilibrium, the buyer chooses not to reveal his type and he stays in store \(i\) unless his expected payoff of doing the opposite exceeds \(1 - \alpha_b\).

Let \(z_b^t\) denote the probability that the buyer is the behavioral type after the history \(h_t\). By Bayes’ law, it is either zero or no less than \(z_b\). By our assumption on \(h_t\), we have \(z_b^t > 0\). Thus, it should be true that \(z_b^t \geq z_b\).

I first show that the game ends (by seller \(i\)’s acceptance of the buyer’s offer \(\alpha_b\)) with probability 1 in finite time, given history \(h_t\), if the buyer continues to stay in store \(i\) and mimics the behavioral type. Finally, I show that as players make offers frequent enough \((\epsilon \to 0)\), the game ends immediately with (almost) no delay.

To see the former claim, note that seller \(i\) can always get payoff at least \(\alpha_b z_b^t\) by seeking almost immediate agreement with the behavioral buyer. On the other hand, if seller \(i\) uses any strategy that could extend the game until period \(t + \hat{t}\) with positive probability (by neither accepting \(\alpha_b\) nor making an offer in which the buyer gets at least \(1 - \alpha_b\)), then seller \(i\)’s expected payoff is at most \(1 - z_b^t + z_b^t e^{-r_i} \alpha_b\).

Thus such a strategy can be optimal for seller \(i\) only if

\[
1 - z_b^t + z_b^t e^{-r_i} \alpha_b \geq \alpha_b z_b^t.
\]

That is

\[
z_b^t \leq \frac{1}{1 + \alpha_b (1 - e^{-r_i})} := \rho
\]

Conditional on the game not ending until time \(t + \hat{t}\) (if the buyer continues to mimic the behavioral type), we can repeat the above argument to conclude that \(z_b^t \leq \rho^2\) for seller \(i\) to optimally follow a strategy at time \(t\) which will not concede to the behavioral buyer until time \(t + 2\hat{t}\).\(^{28}\) Similarly, for the game to last until time \(t + k\hat{t}\), it must be that \(z_b^t \leq \rho^k\). But since \(\rho^k\)

\(^{28}\)Suppose that \(\pi^t_{\hat{t}}\) denotes the probability that the buyer (by rational choice according to his equilibrium strategy) will not observably deviate from the strategy of mimicking the behavioral type from period \(t\) to \(t + \hat{t}\). This implies that \(\pi^t_{\hat{t}} \geq z_b^t\). Conditional on the game not ending until time \(t + 2\hat{t}\), we
converges to zero as \( k \to \infty \) and \( z_b^t \geq z_b \), there will come a \( k^* \) for which this inequality cannot be satisfied. Thus, seller \( i \) must end the game against the buyer who behaves as if he is the behavioral type by time \( t + k^* \hat{t} \).

This argument applies to any game \( G(N,g) \) and shows that the game ends in finite time \( t + t^*(\epsilon) \) if the buyer continues to stay in store \( i \) and mimics the behavioral type.

I will now show that \( t^*(\epsilon) \) converges to zero as \( \epsilon \to 0 \). If this assertion is false then there exist a subsequence of \( \epsilon_n \)'s, an \( \xi > 0 \), a collection of histories \( h_{\epsilon_n} \), and \( t^*(\epsilon_n) \)'s such that game \( G(N,g_{\epsilon_n}) \) conditional on the history \( h_{\epsilon_n} \) ends at time \( t^*(\epsilon_n) \) where \( t^*(\epsilon_n) > \xi \). To simplify the subsequent notation, I will rescale the units of time so that \( r_b = 1 \) and \( r_i = r \).

Consider the last \( \epsilon \)-time units of the game if the buyer continues to mimic the behavioral type. It must be that seller \( i \) is using some strategy (with positive probability) that does not end the game for at least \( \epsilon \) longer.

Let \( x \) be seller \( i \)'s expected payoff if the buyer agrees to an offer worse than \( \alpha_b \) by time \( \beta \epsilon \) for \( \beta \in (0,1) \). Let \( y \) be the seller’s payoff if the buyer does not agree to such an offer by time \( \beta \epsilon \), and let \( \mu \) be the probability that the seller assigns to the event that the buyer will not agree to such an offer by time \( \beta \epsilon \).

Now, the seller’s rejection of \( \alpha_b \) implies that
\[
\mu y + (1 - \mu)x \geq \alpha_b \tag{10}
\]
if and only if
\[
\mu \leq \frac{x - \alpha_b}{x - y} \quad \text{whenever} \quad x > y \tag{11}
\]

Note that for the buyer to agree to a payoff less than \( 1 - \alpha_b \), he must be rational. However, the buyer knows that if he holds out for \( \epsilon \) longer he will get \( 1 - \alpha_b \). Therefore, we must have
\[
x \leq 1 - e^{-\epsilon}(1 - \alpha_b)
\]

Similarly, if the buyer does not agree to an offer by \( \beta \epsilon \), then the best that seller \( i \) can achieve have \( \pi^2_i \geq z_b^t \). Similar arguments yield that \( z_b^t \leq \rho \). Then, \( \pi^2_t = \pi_t^i \pi^2_i \), implying that \( \pi^2_t \geq z_b^t z_b^t \). Since \( z_b^t \) is the probability that the buyer is the behavioral type after the history \( h_t \) and it is the lower bound for the value of \( \pi^2_t \), we must have that \( z_b^t \leq \rho^2 \).

This argument is true for all \( \beta \in (0,1) \) because, if conceding today would be more beneficial than extending the game some time \( \hat{\beta} \epsilon \), then seller \( i \) would have finished the game before reaching that point. Yet we assumed that the game continues for some additional \((1 - \hat{\beta})\epsilon \) amount of time.
after that time is $1 - e^{-(1-\beta)\hat{\epsilon}}(1 - \alpha_b)$. Hence, we must have

$$y \leq e^{-\beta\hat{\epsilon}} \left(1 - e^{-(1-\beta)\hat{\epsilon}}(1 - \alpha_b)\right)$$

Notice that this last inequality implies, for $\hat{\epsilon}$ small enough, that $y \leq \alpha_b$ whenever

$$\beta > \frac{1 - \alpha_b}{1 - \alpha_b + r\alpha_b}.$$  

To see this, note that $e^{-\beta\hat{\epsilon}} \left(1 - e^{-(1-\beta)\hat{\epsilon}}(1 - \alpha_b)\right) < \alpha_b$ if and only if

$$\alpha_b > \frac{e^{-\beta\hat{\epsilon}} \left(1 - e^{-(1-\beta)\hat{\epsilon}}\right)}{1 - e^{-\beta\hat{\epsilon}-(1-\beta)\hat{\epsilon}}}$$

By l'Hopital’s rule, this inequality holds for all $\hat{\epsilon} \in (0, \bar{\epsilon})$, for some $\bar{\epsilon} > 0$ small enough, whenever

$$\beta > \frac{1 - \alpha_b}{1 - \alpha_b + r\alpha_b} \quad (12)$$

Hence, for the values of $\beta$ satisfying the last inequality, (10) holds only if $x \geq \alpha_b > y$. Along with this, the inequality (11) implies that

$$\mu \leq \frac{1 - e^{-\hat{\epsilon}(1 - \alpha_b)} - \alpha_b}{1 - e^{-\hat{\epsilon}(1 - \alpha_b)} - e^{-\beta\hat{\epsilon}}(1 - e^{-(1-\beta)\hat{\epsilon}}(1 - \alpha_b))} = \hat{\rho} < 1$$

Thus, in the final $\hat{\epsilon}$ amount of time, the probability that the buyer will continue to behave like the behavioral type during the first $\beta$ percentage of the time must be less than $\hat{\rho}$. But after $\beta\hat{\epsilon}$ time elapses, the same argument may be repeated to show that at time $t$ the probability that the buyer will continue to behave as if he is the behavioral type until the final $(1 - \beta)^2\hat{\epsilon}$ amount of time must be less than $\hat{\rho}^2$.

Similarly, the probability that the buyer is resisting until the final $(1 - \beta)^k\hat{\epsilon}$ amount of time must be less than $\hat{\rho}^k$. Choosing $k$ such that $\hat{\rho}^k < z_b$ establishes a contradiction since $z_b \geq z_b$ as argued earlier. This argument relies on seller $i$ being able to make offers sufficiently close to time $t + [1 - (1-\beta)^m] \hat{\epsilon}$ for $m = 1, 2, \cdots, k$. Thus, we need the requirement that $g_\epsilon$ converges to its continuous-time counterpart (i.e. $\epsilon$ converges to zero).

Before presenting the proof of Proposition 4.2, I prove two Lemmas that I use extensively later:

**Lemma 4.1.** Let $\epsilon \to 0$ and let $h_t$ be a history such that the buyer is in store $i \in \{1, 2\}$, known to be rational, seller $i$ is unknown to be rational and seller $j \in \{1, 2\}$, $j \neq i$ is known to be the behavioral type. Then, for any sequential equilibrium of the game $G(N,g_\epsilon)$ after the history $h_t$, the payoff to the buyer is no more than $1 - \alpha + \epsilon$ and the payoff to the seller $i$ is no less than $\alpha - \epsilon$ (payoffs are evaluated at time $t$).
Proof. Given that seller $j$ is the behavioral type, the buyer’s continuation payoff in store $j$ is at most $1 - \alpha$. Therefore, the buyer has no incentive to leave store $i$ to get a price better than $\alpha$. Given this, seller $i$ does not reveal his type unless he gets a payoff higher than $\alpha$ by doing the opposite. Hence, the payoff to the buyer is no more than $1 - \alpha$ as $\epsilon$ converges to zero. \qed

Lemma 4.2. Let $\epsilon$ converge to 0 and let $h_t$ be a history such that the buyer is in store $i \in \{1, 2\}$ and known to be rational while both sellers are unknown to be rational. Then in any sequential equilibrium after the history $h_t$ it cannot be the case that seller $i$ finishes the game at time $t$ at some price $x < \alpha - \epsilon$ with probability one.

Proof. Suppose for a contradiction that rational seller $i$ makes a deal with the buyer at some price $x < \alpha$ at time $t$ with probability 1. Given that this is an equilibrium strategy, both seller $j$ and the buyer assign probability 1 to the event that seller $i$ is the behavioral type if the seller does not accept the buyer’s offer. But then, according to Lemma 4.1, the buyer accepts the price $\alpha$ and finishes the game immediately at time $t^*$ where $t < t^* \leq t + \epsilon$.

However, for arbitrarily small $\epsilon$, rational seller $i$ would prefer to deviate from his equilibrium strategy and wait until time $t^*$ by mimicking the behavioral type so that he can get the payoff of $\alpha$ which is higher than $x$.\footnote{Receiving $\alpha$ at time $t^*$ is equivalent to receiving $\alpha e^{-r_i(t^*-t)}$ at time $t$, which is arbitrarily close to $\alpha$ as $\epsilon$ converges to 0.} Hence, in equilibrium after the history $h_t$ seller $i$ delays the game with a positive probability. \qed

Proof of Proposition 4.2. Without loss of generality, suppose that the buyer is in store 1 at time $t$ after the history $h_t$. I will show that as seller 1 continues to mimic the behavioral type, the payoff to the buyer converges to $1 - \alpha$ and the payoff to seller 1 converges to $\alpha$, as $\epsilon$ converges to zero. For the remainder of this proof, assume that seller 1 continues to mimic the behavioral type, while the buyer and seller 2 execute their equilibrium strategies.

For $i \in \{1, 2\}$, let $z_i^t$ denote the probability that seller $i$ is the behavioral type at time $t$ after the history $h_t$. By Bayes’ rule, $z_i^t$ is either zero or higher than $z_i$. By our assumption, however, for each $i$, we must have $z_i^t \geq z_i$.

If the buyer continues to stay in store 1 for long enough according to his equilibrium strategy while seller 1 continues to act irrationally, we know by proposition 4.1 that the payoff to the buyer converges to $1 - \alpha$ as $\epsilon$ converges to zero, and this proves the claim of the proposition. It is, however, possible that in equilibrium the buyer does not stay in store 1 long enough if seller
continues to mimic the behavioral type. This implies that the buyer leaves store 1 at some time \( t' \geq t \). Note that \( z_2^t = z_2^{t'} \).

The buyer’s decision of leaving store 1 at time \( t' \) implies that
\[
z_2^{t'} \leq \frac{\delta + \alpha - 1}{\delta \alpha} = \hat{\rho} < 1
\]
This is true because, if the buyer goes to store 2 and seeks an agreement with seller 2, the highest payoff he could achieve is \( \delta[1 - z_2^t + (1 - \alpha)z_2^t] \). But leaving store 1 and going to store 2 at time \( t' \) is optimal for the buyer only if
\[
1 - \alpha \leq \delta[1 - z_2^t + (1 - \alpha)z_2^t]
\]
which implies the desired result.

According to his strategy, if the buyer continues to stay in store 2 long enough, conditional on seller 2 mimicking the behavioral type, we know again by Proposition 4.1 that the payoff to the buyer converges to \( 1 - \alpha \) as \( \epsilon \) converges to zero. This implies that \( 1 - \alpha \) is the highest payoff the buyer can attain in store 2. If this is the case, however, the buyer does not leave store 1 at time \( t' \), which contradicts our supposition. Therefore, it must be the case that the buyer leaves store 2 as well, conditional on seller 2 continuing to mimic the behavioral type, at some time \( t'' \) where \( t'' > t' \).

According to his equilibrium strategy, seller 2 may be playing a strategy that ends the game while the buyer is in store 2. However, according to Lemma 4.2, we know that seller 2 will not play a strategy that will end the game with a price less than \( \alpha \) (in the limit) with probability one. If rational seller 2 is playing a strategy which ends the game with a price higher than \( \alpha \), then he buyer does not leave store 1 at time \( t' \), which contradicts our supposition. Therefore, it must be the case that seller 2 is playing a strategy that extends the game, i.e. seller 2 will mimic the behavioral type with a positive probability, until time \( t'' \).

Conditional on the buyer arriving to store 1 once more, the same arguments show that the buyer shall leave store 1 once again as seller 1 continues to mimic the behavioral type (because otherwise, the payoff to the buyer will be at most \( 1 - \alpha \) and this contradicts our supposition that the buyer leaves store 2 when seller 2 continues to mimic the behavioral type).

Therefore, conditional on both sellers extending the game and the buyer leaving store 1 twice, we have \( z_2^t \leq \hat{\rho}^2 \), so that extending the game by going back and forth between the sellers (twice) is more profitable for the buyer than seeking an immediate agreement with irrationally behaving seller 1.
Similarly, for the game that lasts until the $k^{th}$ departure of the buyer, it must be true that $z_t^2 \leq \hat{\rho}^k$. Choosing $k$ such that $\hat{\rho}^k < z_2$ establishes contradiction since, as argued earlier, $z_t^2 \geq z_2$.

Therefore, as seller 1 continues to mimic the behavioral type, seller 2 will continue to play a strategy which extends the game with positive probability (immediate consequence of Lemma 4.2). The buyer, however, will travel back and forth between the sellers only for some finite time in order to get a deal better than $\alpha$. This implies that the buyer will end up at some store $i \in \{1, 2\}$ at some finite time $\bar{t}$. That is, the buyer does not leave store $i$ after time $\bar{t}$ while seller $i$ continues to mimic the behavioral type.

This implies that the buyer’s continuation payoff in store $i$ is at most $1 - \alpha$, evaluated at time $\bar{t}$. This leads to a contradiction because, given that the buyer’s continuation payoff in his final destination is less than $1 - \alpha$, the buyer should not have left store $j$ when seller $j$ continues to act irrationally. Hence, repeating this argument backward, we can conclude that the buyer does not delay the game, but instead seeks an immediate agreement with seller 1 at time $t$.

**Proof of Proposition 4.3.** This proof is adapted from the proof of Proposition 4 in Abreu and Gul (2000). Propositions 4.1 and 4.2 imply that once a player reveals that he is not the behavioral type, agreement must be reached almost immediately, at terms arbitrarily close to the demand of his still possibly behavioral opponent. Therefore, as in the continuous-time game, we can identify revealing rationality with conceding to one’s opponent’s irrational demand. I first show convergence of post revelation equilibrium payoffs, which then implies the convergence of the overall equilibrium along with Propositions 1B and 2B.

Let $G(N, g_{\epsilon n})$ be a sequence of discrete-time bargaining problems and $\sigma^n$ (drop the term $\epsilon$ to ease the notation) be the corresponding sequence of sequential equilibria. For each $\sigma^n$, let $\sigma^n_b$ denote the buyer’s strategy that determines his location at any given history. Then, given a history $h_t$, define $S_t^{\sigma^n_b}$ and $I_t^{\sigma^n_b}$ as in the proof of Proposition 2.1. Then, for each $i \in \{1, 2\}$ and $T \in I_t^{\sigma^n_b}$ with $[T, T'] \in S_t^{\sigma^n_b}$ define $F_{n,T}^{i}: [T, T'] \rightarrow [0, 1]$, where $F_{n,T}^{i}(t)$ is the cumulative probability that seller $i$ takes an action not consistent with the behavioral type in the interval $[T, t]$, conditional on the buyer and the other seller having acted like a behavioral type until time $t$.

Similarly, define $F_{n,T}^{b,i}: [T, T'] \rightarrow [0, 1]$ where $F_{n,T}^{b,i}(t)$ is the cumulative probability that the buyer takes an action not consistent with the behavioral type in the interval $[T, t]$, conditional on the buyer and the other seller having acted like a behavioral type until time $t$.

For the sake of simplicity, I manipulate the notation and denote the continuation strategy following the history $h_t$ by $\sigma^n_b$. 

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on the buyer is in store \( i \) in this time interval according to \( \sigma^n_i \) and both sellers having acted as if they are behavioral until time \( t \).

To prove the Proposition, arbitrarily choose some \( \bar{n} \geq 0 \), an equilibrium strategy \( \sigma^\bar{n} \) and a history \( h_t \) to fix the sets \( S^{\bar{n}}_i \) and \( T^{\bar{n}} \). Then, for each \( i \in \{1, 2\} \) and \( T \in T_i^{\bar{n}} \) with \( [T, T'] \in S^{\bar{n}}_i \), I show that as \( n \geq \bar{n} \), (1) every subsequence of \( F^{i,T}_n \) and \( F^{b,i,T}_n \) have a convergent subsequence; (2) the limit points of \( (F^{i,T}_n, F^{b,i,T}_n) \) do not have common points of discontinuity in the domain \( [T, T'] \); (3) if \( (F^{i,T}_n, F^{b,i,T}_n) \) converges to \( (F^i_T, F^{i,T}_b) \) and if the limit functions do not have common points of discontinuity then \( (F^i_T, F^{i,T}_b) \) is an equilibrium of the continuous-time bargaining problem in the interval \( [T, T'] \).

Thus, (1) – (3) imply that for each \( i \in \{1, 2\} \), and \( T \in T_i^{\bar{n}} \) with \( [T, T'] \in S^{\bar{n}}_i \), \( (F^{i,T}_n, F^{b,i,T}_n) \) converges to the equilibrium distribution of the continuous-time game. Since the equilibrium distribution within each interval is unique (by Proposition 2.1), we conclude that the limit of \( (F^{i,T}_n, F^{b,i,T}_n) \) is equal to \( (F^i_T, F^{i,T}_b) \), the unique equilibrium distribution of the continuous-time bargaining problem analyzed in Section 2. Also, by Propositions 4.1 and 4.2 we conclude that \( \theta_e(\sigma^0) \) converges in distribution to the equilibrium outcome of the continuous-time game \( \theta(\sigma^0) \).

The following two steps establish that every subsequence of \( (F^{i,T}_n, F^{b,i,T}_n) \) has a convergent subsequence.

**Step 1** There exist finite \( N \) and \( T^* \in T_i^{\bar{n}} \) such that \( F^{i,T}_n(T^*) = 1 - z^T_i \) for all \( n \geq N \). That is, a rational player must concede in finite time \( T^* \) to an opponent who persists in irrational behavior. This argument is identical to the proof of Propositions 1B and 2B.

Therefore, the game ends at some finite time \( T^* \). So, for any \( T \in T_i^{\bar{n}} \) and \( [T, T'] \in S^{\bar{n}}_i \) where \( T < T^* \), define \( G^i_n \) such that

\[
G^{i,T}_n(t) = \frac{F^{i,T}_n(t)}{F^{i,T}_n(T')}
\]

whenever \( F^{i,T}_n(T') \neq 0 \).

for all \( t \leq T' \) where \( F^{i,T}_n(T') = 1 - z^T_i / z^{T'}_i \). The last argument is due to the fact that \( z^T_i \) is updated according to Bayes’ rule, i.e. \( z^{T'}_i = \frac{z^T_i}{1-F^{i,T}_n(T')} \). Note that \( \{F^{i,T}_n(T')\} \) is a bounded real sequence, which is bounded below by 0 and above by \( 1 - z_i \) for all \( n \). The same arguments hold for the buyer.

**Step 2** There exists a subsequence \( \{F^{i,T}_{n_k}\} \) and a nondecreasing right continuous function \( F^T_i \) such that \( \lim_k F^{i,T}_{n_k}(t) = F^T_i(t) \) at the continuity points \( t \) of \( F^T_i \). Furthermore, \( F^T_i(t) \leq 1 - z^T_i \) for all \( t \geq T \). The same is true for the buyer.
Proof. Note that \( G_{ni,T}^i(t) = 1 \) for all \( n \) and \( t \geq T \) establishes that \( G_{ni,T}^i \) is a distribution function. Therefore, by Helly's selection Theorem (See Billingsley (1986)), the sequence \( G_{ni,T}^i \) has a subsequence \( G_{ni,T}^{i,k} \) which converges to a right continuous, non-decreasing function \( G_{T}^{i} \) at every continuity point of \( G_{T}^{i} \).

Now, let \( F_{n,k}^{i,T} = F_{n,k}^{i,T}(T')G_{n,k}^{i,T} \). Since the real sequence \( F_{n,k}^{i,T}(T') \) is bounded below 0 and bounded above 1 - \( z_i \) for any \( n \), there must exist a subsequence \( F_{n,k_j}^{i,T}(T') = F_{n,k_j}^{i,T}(T')G_{n,k_j}^{i,T} \) implies that \( F_{n,k_j}^{i,T}(T') = F_{n,k_j}^{i,T}(T')G_{n,k_j}^{i,T} \). Apply the same arguments to the buyer and renumber the sequence \( n_{k_j} \) will yield the desired result.

For the remaining part of the proof, renumber the sequence so that \( (F_{n,k_j}^{i,T}, F_{n,k_j}^{b,i,T}) \) may be denoted by \( (F_{n}^{i,T}, F_{n}^{b,i,T}) \).

Next I establish that the limit of \( (F_{n}^{i,T}, F_{n}^{b,i,T}) \) forms an equilibrium of the continuous time game in the interval \([T,T']\). The proofs of the following claims utilize the exact methods used in the proof of steps 3-6 in Abreu and Gul (2000). Therefore, I do not represent their proof to prevent duplication.

Note that the proofs of the following arguments rely on the fact that \( (F_{n}^{i,T}, F_{n}^{b,i,T}) \) is an equilibrium over the interval \([T,T']\) for any \( n \geq \bar{n} \). At some \( n^* > \bar{n} \), the continuation strategy \( \sigma_{b}^{n^*} \) may lead to different sets of \( S_{i}^{b,n^*} \) and \( I_{i}^{b,n^*} \). However, sequential rationality implies that \( (F_{n^*}^{i,T}, F_{n^*}^{b,i,T}) \) is an equilibrium over the interval \([T,T']\).

**Step 3** \( (F_{T}^{i,T}, F_{b}^{i,T}) \) have no common discontinuity points: Suppose that at some time \( t \in [T,T'] \) both functions have positive jump. Choose \( \Delta > 0 \) small enough so that both \( t - \Delta \) and \( t + \Delta \) are continuity points of \( F_{T}^{i,T} \) and \( F_{b}^{i,T} \). Then, given that a player concedes with positive probability at time \( t \), for large enough \( n \), his opponent prefers to wait until \( t + \Delta \). This tendency continues in the limit, contradicting the presumption.

**Step 4** Let \( h : \mathbb{R}^2 \to \mathbb{R} \) be a Lebesgue measurable function and let \( D_h \) denote the set of discontinuity points of \( h \). If \( \mu_n \) is a sequence of probability measures converging in distribution to some probability measure \( \mu \) where \( \mu \) has bounded support, then \( \mu_n h^{-1} \) converges in distribution to \( \mu h^{-1} \) whenever \( \mu(D_h) = 0 \). Hence, \( \int h d\mu_n \) converges to \( \int h d\mu \).

**Step 5** If \( G_{ni,T}^{i} \) converges in distribution to \( G_{iT}^{i} \) and \( G_{ni,T}^{b,i} \) converges in distribution to \( G_{bi,T}^{i} \), then \( \mu_n \) the product measure (on \( \mathbb{R}^2 \)) associated with \( G_{ni,T}^{i}, G_{ni,T}^{b,i} \), converges to \( \mu \), the product measure associated with \( G_{iT}^{i} \) and \( G_{bi,T}^{i} \).
Step 6 If $G_n^{i,T}$ and $G_n^{b,i,T}$ converge to $G_i^T$ and $G_b^{i,T}$ respectively, $G_n^{i,T}(t) = G_i^T(t) = G_n^{b,i,T} = G_b^T = 0$ for all $t < T$ and equal to 1 for all $t \geq T$ and $G_i^T, G_b^{i,T}$ have no common points of discontinuity, then

$$\lim U^i(F_n^{i,T}(T';G_n^{i,T},F_n^{b,i,T}(T';G_n^{b,i,T})) = (F_i^T(T')G_i^T,F_b^{i,T}(T')G_b^T)$$

where the utility functions have been defined in section 2: Define

$$h^i(t_i,t_b) = \begin{cases} \alpha_b & \text{if } t_i < t_b \\ \alpha & \text{if } t_i > t_b \\ \frac{\alpha+\alpha_b}{2} & \text{if } t_i = t_b \end{cases}$$

Since $G_n^{i,T}$ and $G_n^{b,i,T}$ converge in probability to $G_i^T$ and $G_b^{i,T}$ respectively, step 5 implies that $G_n^{i,T} \times G_n^{b,i,T}$, the product measure, converges to $G_i^T \times G_b^{i,T}$. Since $G_i^T$ and $G_b^{i,T}$ have no common points of discontinuity, the set $D_b := \{(t_i,t_b)|t_i = t_b\}$ has zero $G_i^T \times G_b^{i,T}$ measure. Therefore, step 4 yields the desired result.

Step 6 implies that if $t_n \to t$, and if $G_n^{b,i,T}$ is continuous at $t$, then

$$\lim U^i(t_n,F_n^{b,i,T}(T';G_n^{b,i,T})) = U^i(t,F_b^{i,T}(T')G_b^{i,T}).$$

Similarly, if $G_i^T$ is continuous at $t$, then $\lim U^b(F_n^{i,T}(T')G_n^{i,T},t_n) = U^b(F_i^T(T')G_i^T,t)$. Recall that the arguments $\tau = t, t_n$ are shorthand for the degenerate strategy in which the rational type concedes with probability one at time $\tau$.

The following arguments complete the proof of our claim regarding convergence within each interval $[T, T'] \in S_i^\sigma$. Recall that $\sigma^i$ is the equilibrium strategy of the game $g_n$. For any $t > 0$ and $\varepsilon > 0$ define a strategy $\bar{\sigma}_n^i$ to be a strategy of seller $i$ within the interval $[T, T']$ as follows: Seller $i$ behaves according to $\sigma_n^i$ until time $t_n$ where $t_n$ is the last time the buyer makes an offer prior to $t + \varepsilon$ (for some $\varepsilon > 0$) and at time $t_n$ seller $i$ accepts the buyer’s offer $\alpha_b$. Let $U_n^i$ denote the utility function of seller $i$ in the game $g_n$. Then there exist finite integers $N_1, N_2, N_3$ and $\varepsilon > 0$ sufficiently close to 0, such that $t + \varepsilon$ is a continuity point of $F_b^{i,T}$ and

$$U^i(t,F_b^{i,T}) - U^i(t+\varepsilon,F_b^{i,T}) < \varepsilon, \quad (13)$$

$$U^i(t+\varepsilon,F_b^{i,T}) - U^i(t_n,F_n^{b,i,T}) < \varepsilon \quad \forall n \geq N_1, \quad (14)$$

$$U^i(t_n,F_n^{b,i,T}) - U^i(\bar{\sigma}_n^i,\sigma_n^b) < \varepsilon \quad \forall n \geq N_2, \quad (15)$$

$$U_n^i(\bar{\sigma}_n^i,\sigma_n^b) - U_n^i(\sigma_n^i,\sigma_n^b) \leq 0 \quad \forall n, \quad (16)$$

$$U_n^i(\sigma_n^i,\sigma_n^b) - U_n^i(F_n^{i,T},F_n^{b,i,T}) < \varepsilon \quad \forall n \geq N_2, \quad (17)$$

$$U^i(F_n^{i,T},F_n^{b,i,T}) - U^i(F_i^T,F_b^{i,T}) < \varepsilon \quad \forall n \geq N_3. \quad (18)$$
Equation (13) follows immediately from the definition of $U_i$. That is, $U_i(.,F^i_{b,T})$ is continuous at continuity points of $F^i_{b,T}$. If $t$ is not a continuity point of $F^i_{b,T}$, for $\hat{\epsilon}$ small enough the left-hand side of (13) is strictly negative (similar logic to the proof of step 3: if the buyer makes a mass acceptance at time $t$, seller $i$ would prefer conceding at time $t + \hat{\epsilon}$ over conceding at time $t$). Since $t + \hat{\epsilon}$ is a continuity point of $F^i_{b,T}$, (14) follows from Step 6. Equation (15) follows from the definition of $\tilde{\sigma}_n$ and Proposition 4.2. Equation (16) is the consequence of the fact that $(\sigma^i_n, \sigma^b_n)$ is equilibrium. Equation (17) is an application of Proposition 4.1; seller $i$ can never get more than $\alpha_b$ after revealing his rationality. Moreover, in equilibrium, since his opponent makes offers frequently, he can reveal himself to be rational in a manner that guarantees $\alpha_b$. Equation (18) follows from Steps 3-6.

Choosing $n \geq \max\{N_1, N_2, N_3\}$ and adding Equations (13)-(18) will yield

$$U^i(t,F^b_{T,i,T}) - U^i(F^i_{i,T},F^b_{b,i,T}) < 5\hat{\epsilon}$$

Since this inequality is true for any $\hat{\epsilon} > 0$, it must be the case that

$$U^i(t,F^b_{T,i,T}) - U^i(F^i_{i,T},F^b_{b,i,T}) \leq 0.$$  

Hence, $F^b_{T,i}$ is a best response to $F^i_{b,T}$. Symmetric arguments imply that $(F^i_{T,i},F^b_{b,T})$ is a Nash equilibrium of the continuous-time game within the interval $[T,T']$. Note that if seller $i$ is the first to reveal his type, he can guarantee $\alpha_b$ by accepting the buyer’s offer. This would yield the buyer a payoff of $1 - \alpha_b$. If seller $i$ reveals his type in some other way, then by Proposition 4.1 he is still, in the limit, guaranteed $\alpha_b$. This happens only if agreement is reached immediately at these terms. Analogous arguments are valid for the buyer. Therefore, convergence in expected payoffs implies convergence in distribution within the interval $[T,T']$.

After an arbitrary history $h_t$ and continuation strategy $\sigma^a_n$, I proved the convergence in each interval $[T,T'] \in S^a_n$. So, for given $\sigma^a_n$ let $F_n$, the distribution function profile of the discrete-time bargaining problem $G(N,g_{t_n})$, converge to $F$, the distribution function profile of the continuous-time bargaining problem $G$, history by history (i.e., interval by interval) in the product topology.

Given that $F_n$ converges to $F$ history by history, similar arguments in the proof of Proposition 2.3 suffice to show that for sufficiently large $n$, the buyer visits each store at most once according to the equilibrium strategy of the game $G(N,g_{t_n})$. As a result, convergence in distribution in all subgames implies that the buyer’s timing and location decisions together with the distribution functions, $F_n$, converge to the unique (up to the buyer’s store selection at time 0) equilibrium of the continuous-time problem.
References


