Economics of Open Source Technology: A Dynamic Approach*

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Abstract

We analyze open source licensing and its effects on firms' decisions whether to use the open source technology or not and on the incentives for innovation, through a dynamic model of innovation and competition in an environment with a ladder type technology. We model the basic features of the General Public License (GPL), one of the most popular open source licenses and study how firms behave under this license when competition is present. Under the GPL, any innovative findings using open source technology must also be open source in the subsequent periods, and this obligation creates a trade-off. We focus on how this trade-off affects incentives to use and build up the open source technology.

Keywords : open source technology, General Public License, oligopolistic competition, innovation.

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1 Introduction

Open software development involves a major deviation from the private investment model of innovation; open source innovators freely share the proprietary software that they have developed at their private expense. For example, Linux, a computer operating system, is evolving with many independent developers revealing the code to develop and refine it. Its source code is *open* in the sense that anyone has free access to it. One of the most popular web servers has always been an open source software. [1]

The success of open source software raises many questions about innovation policies with non-traditional property rights.[2] We particularly pay attention to the fact that, although the source code of open source software is freely available, open source programs are distributed under very precise licensing agreements. The broad purpose of this paper is to improve our understanding on how a certain form of license affects firms' incentives of both innovation and participation in an open source community. Specifically, we provide a dynamic model that captures the important characteristics of open source licensing and explain, to some extent, the open source development phenomenon. To capture important features open source innovation and its licensing have, we focus on a concrete example, the GNU General Public License (GPL) which is one of the most common licenses.[3] Among a number of features GPL has, we can summarize some of the crucial ones as follows. First, while every user has the freedom to use and modify programs subject to the GPL, such modifications must be distributed under the terms of the license itself if they are to be distributed at all. Second, the GPL does not preclude the commercial exploitation of the software, at any stage. That is, the program users have to maintain the free access to the source, but they do not need to share any profit they make. It is evident that there is strong competition in this field. Once open source code is *improved* by a firm, by its nature, it is accessible to its customers or even to its competitors. However, due to its complexity of programming, the *inventor* can enjoy advantageous position as the first mover for a span of time.

In this paper, we study these major features of open innovation and their effects on incentives for innovation and usage of the open source technology. In order to capture these key characteristics, namely the dynamic nature of the open source development and the GPL restrictions (successful

[1] For example, it is well known that *Apache*, an open source software web server has been dominating the public Internet web server market ever since April 1996. Another successful open source software example might be *Perl*, a robust scripting language which is used for high-traffic websites such as *Priceline.com*, *Ticketmaster*, and *IMDb*.

[2] See Lerner and Tirole (2002) for a broad discussion of issues concerning economics of open source technology.

[3] GNU is a recursive acronym which stands for "GNU's Not Unix".

innovations being subject to GPL) with allowance of open source technology firms making profit, we use a T-period 3-stage model where, in each period, firms decide whether to use the open source technology in the first stage, pursue technology advancement through cost-reducing investment in the second stage, and engage in Cournot competition in the third stage. Within the framework of our model, which we believe reflects these important features of the open source innovation in a more direct way than the existing literature does, we study the open source technology usage decisions of the firms as well as their investment decisions in innovation and competition quantities. We also extend the model into an infinite horizon case.

We characterize when it is optimal for a firm to join the open source community and when it is not, together with the investments in innovation. The main tradeoff in our analysis is stemming from the very nature of the GPL licensing. When a firm is deciding to join the open source community, it will be able to use the open source technology at no direct cost, and this will potentially remove a cost disadvantage in the competition stage against other firms. However, if this firm, after joining the open source community, succeeds in innovation or in cost reduction, then in later periods it has to make it public due to licensing, and thus, might lose a potential cost advantage, which may not be the case if instead it stayed out of the open source community. This may decrease the expected future revenue of the firm, when joined to the open source community. On the other hand there is also a direct benefit of joining the open source community. We investigate how this trade-off influences firms' open source technology use decision depending on their technology level relative to the open source technology level. We find that this tradeoff is resolved in a way that when a firm is at the same level with (or at a higher level than) the open source technology, it does not prefer to join the open source community. If, however, it is behind the open source in the technology ladder, it optimally chooses to use the open source. Thus, if the open source community succeeds consistently in innovation (cost reduction), then the open source community will sweep out the proprietary firms, otherwise there will be a set of firms at a higher level than the open source technology.

The literature on economics of open source technology has been growing since early 2000s. Lerner and Tirole (2002, 2005) provide a general discussion of the economics of open source development and lay out a broad literature review. They point out that the open source developers receive a direct effect in the form of improved open source technology, since they directly benefit from it, and an indirect effect through signalling their abilities and through reputational gains. They show that the literature mostly considers individual motives, incentives to adopt open source softwares and the effect of competition within an open source community.[4] The main contribution of our study is to present a simple model which is both dynamic and tractable, and it is capable of capturing the essence of the GPL licensing.

A good amount of the literature focuses on the open source development as a public good and uses a static approach (e.g., Johnson 2002, Llanes and Elejalde 2013, Atal and Shankar 2014). Our model and approach differs from these studies in the way it captures the essence of the GPL licensing in open source innovation environments. Since these studies use a static model, they do not fully capture the main characteristic of the GPL, which is basically "get it for free *now*, pay back *when/if* you succeed." We believe, our model captures the essence of the GPL in a more direct way, when a firm uses the open source technology, then its successful innovation is made available in later periods, yet in the current period with the innovation success, the firm can enjoy an advantage in the competition stage.

Among those studies which use a dynamic approach, Athey and Ellison (2014) use a dynamic model where the open source user/programmers are motivated by reciprocal altruism. The evolution of the open source technology depends on the quality and the altruistic developers. Bitzer, Schrettl and Schröder (2007) provide a dynamic model of private provision of a public good and focus on the intrinsic motivation of the programmers to explain the open source development. We are not modelling the firms/developers as altruistic or intrinsically motivated players, rather they are strategic agents who wish to maximize expected profits.

Finally, among firms' decision problems related to open source technology, we focus on whether a firm joins an available open source community or not. Caulkins et al. (2013) study the question of how long does a firm keep its software proprietary and when does it release it to be open source technology, over a continuous time dynamic model where firms invest in quality and pick own price for its software and complementary product. Kort and Zaccour (2011) study a similar problem through a 3 stage duopoly game, where they characterize the conditions under which it is optimal for a firm to open its code.[5] Though these questions are also important, our model addresses a more fundamental question, thus may complement their findings.

Section 2 depicts the model. Section 3 solves the model and provides the main results for finite horizon and Section 4 extends it to the infinite horizon. In Section 5, we discuss some relevant points and extensions. Section 6 concludes.

[4] See Lerner, Pathak and Tirole (2006) for an empirical study on the dynamics of contributions to open source software projects.

[5] Also see Haruvy, Sethi and Zhou (2008).

2 The Model

There are $M \geq 2$ firms interacting over $T \geq 3$ many periods. Each period t, each firm i produces a good at a firm-period specific unit cost $c(k_i^t)$, which depends on the firm's technology level k_i^t and is stochastically determined by firm i's investment in cost-reducing innovation. There is also a public production technology, called *open source technology*, which can produce the good in period t at a unit cost $c(k_{os}^t) > 0$, which depends on the open source technology level, k_{os}^t, which is also stochastically determined by the open source technology using firms' investments in cost-reducing innovation.

Chain of events within a period: In each period, there are three stages:

(1) each firm decides whether to adopt the open source technology or not,

(2) each firm invests in cost-reducing innovation,

(3) firms compete in quantities in a Cournot fashion.

To capture the effect of open source technology under GPL, we make the following assumptions.

Assumption 1 *Each firm is free to use the open source technology at no direct cost.*

Assumption 2 *Any innovation made by a firm which uses the open source technology in period t, will be open source technology from period $t+1$ on.*

Production cost: To be more precise, let $k_i^t \in \mathbb{Z}_+$ denote the production technology level for firm i at the beginning of period t. The unit cost of firm i is given by the function $c(k_i^t)$, with $c'(\cdot) < 0$ and $lim_{k \to \infty} c(k) = 0$. That is, the unit cost is strictly decreasing as the production technology level increases and in the limit as the technology level k goes to infinity, the unit cost goes to zero. Let $k_{os}^t \in \mathbb{Z}_+$ denote the production technology level of the open source technology at the beginning of period t. That is, before period t starts, the public production technology level is able to produce the good at a unit cost $c(k_{os}^t)$. Likewise, k_{os}^1 is the initial technology level of the open source technology at the beginning of period 1. We keep this general form for the cost function through out the model and through our results.[6]

Open source technology use decisions: At the first stage of each period t, each firm i who has not used the open source technology before decides whether to use the open source technology or not.

[6] In case of open source software, using more advanced or upgraded software may reduce the unit cost by lessening the repetitive efforts to fix bugs in the previous version or by processing the same volume of data at a faster rate than before. Thus, although having a higher technology level may reduce the fixed costs, we also believe that a unit cost function that decreases with the technology level is also plausible. For instance, open source based commercial softwares in the Project Management Software product market such as *Project-open*, *Collabtive*, and *eGroupware* may serve as a good example with such a cost function.

Let $d_i^t \in \{0,1\}$ denote this open source technology use decision of firm i at the first stage of period t, where 1 stands for *use* decision and 0 stands for *not use* decision. We call a firm with $d_i^t = 0$ a *non-user* firm and a firm with $d_i^t = 1$ a *user* firm. When a firm is indifferent between *using* and *not using* the open source technology, we assume that it chooses to use it. If a firm j has already used the open source technology at some period t', then $d_j^t = 1$ for each $t \geq t'$.[7] Also, a firm can freely use the open source technology at no direct cost (Assumption 1 above). Let $\kappa_i^t(d_i, k_i)$ denote the technology level of firm i after its open source technology use decision d_i^t, before which it had a technology level k_i. That is,

$$\kappa_i^t(d_i, k_i) = d_i^t \max\{k_i^t, k_{os}^t\} + \left(1 - d_i^t\right) k_i^t.$$

Note that $\kappa_{os}^t = k_{os}^t$ for each t, since the open source technology has no *use-not use* decision.

Investment in innovation: At the second stage of each period t, each firm i decides how much to invest in innovation. We model investment in innovation through probability of success: firm i with a technology level k_i^t and open source technology use decision d_i^t, picks a probability of success, $p(d_i^t, k_i^t)$,[8] at a cost $C(p(d_i^t, k_i^t))$ with $C' > 0$ and $C'' > 0$. With probability $p(d_i^t, k_i^t)$ there is success and the firm advances one level in the technology ladder, that is, achieves a new level, $\kappa_i^t + 1$, and reduces its unit cost. With probability $1 - p(d_i^t, k_i^t)$ the firm fails to advance one level and stays at the current technology level κ_i. We denote the realization of the new technology level with K_i^t.

If a firm is using the open source technology at period t, then its technology level depends on the open source's current technology level and its own success/failure outcome. More precisely,

$$K_i^t = \begin{cases} \kappa_i^t + 1 & \text{with probability} \quad p(d_i^t, k_i^t) \\ \kappa_i^t & \text{with probability} \quad 1 - p(d_i^t, k_i^t) \end{cases}$$

If $d_i^t = 1$, then

$$K_i^t = \begin{cases} \max\{k_i^t, k_{os}^t\} + 1 & \text{if success} \\ \max\{k_i^t, k_{os}^t\} & \text{if fail} \end{cases}$$

The open source technology level depends on its previous period technology level and the previous period's success/failure outcome of the firms that were using the open source technology. That is, the

[7] Here we assume that a firm who has already joined the open source community cannot leave it. A more general way to model it would be to allow the firms to leave the open source community whenever they want, and show that they will not leave it in the equilibrium. We discuss this in the Section 5.

[8] We write $p(d_i^t, k_i^t)$ instead of $p(\kappa_i^t)$, since two firms with the same κ^t might choose different probabilities if they have different d^t.

successful innovation by a user firm is reflected on the open source technology with exactly one period lag (Assumption 2 above). More precisely,

$$K_{os}^t = k_{os}^t = \begin{cases} k_{os}^{t-1} + 1 & \text{with probability} \quad 1 - [1 - p(1, k_{os}^{t-1})]^{n_1(k_{os}^{t-1}, t)} \\ k_{os}^{t-1} & \text{with probability} \quad [1 - p(1, k_{os}^{t-1})]^{n_1(k_{os}^{t-1}, t)} \end{cases}$$

where $n_1(k_{os}^{t-1}, t-1)$ is the number of firms with $d_i^{t-1} = 1$ in period $t-1$.

Cournot competition: At the third stage of each period t, firms engage in quantity competition a la Cournot. Each firm i simultaneously decides how much to produce, $q_{K_i^t}$, when it has a technology level K_i^t at the beginning of the third stage. A firm with K_i^t has an expected inverse demand given by $\mathbf{P}_{K_i^t} \equiv P(\mathbf{Q}_{K_i^t}) = A - \mathbf{Q}_{K_i^t}$, where $A > 0$ is sufficiently large, P is the market price, and $\mathbf{Q}_{K_i^t}$ is the total quantity demanded that a firm with K_i^t expects. This total quantity can be decomposed into two parts, its own quantity, which is known, and expected total quantity of all other firms, that is, $\mathbf{Q}_{K_i^t} = q_{K_i^t} + \mathbf{Q}_{-i}^t$. At the end of this stage, each firm realizes its profit level $\pi_{K_i}^t$ in the Cournot competition.

Overall payoff of a firm: Firm i's overall payoff is its discounted sum of within period Cournot profits and cost of investment. That is,

$$\Pi_i = \sum_{t=1}^{T} \delta^{t-1}[\pi_{K_i}^t - C(p(d_i^t, k_i^t))]$$

where δ is the discount factor of firm i.

Distribution of firms: At the beginning of the first stage of period $t = 1$, there are $n(k, 1)$ firms with unit cost $c(k)$ where $\sum_k n(k, 1) = M$ and $k \in \{0, 1, 2\}$. Here, 1 in $n(k, 1)$ refers to the first period. We assume that the initial technology level of the open source is $k_{os}^1 = 1$. Thus, there are firms which are at a lower technology level than the open source technology, firms which are at a higher technology level than the open source technology, and firms which are at the same technology level as the open source technology. The number of firms that have the technology level $k_{os}^1 = 1$ at period $t = 1$, $n(1, 1)$, is composed of user and non-user firms: $n(1, 1) = n_0(1, 1) + n_1(1, 1)$, where $n_0(1, 1)$ and $n_1(1, 1)$ denote the number of non-user and user firms, respectively. We assume that each of $n(0, 1)$, $n_0(1, 1)$, $n_1(1, 1)$ and $n(2, 1)$ are publicly observed at the beginning of the first stage.

At the beginning of the first stage in period $t > 1$, there are $n(k, t)$ firms with unit cost $c(k)$ where $k \in \{0, 1, ..., t + 2\}$. For each k, $n(k, t) = n_0(k, t) + n_1(k, t)$, where $n_0(k, t)$ and $n_1(k, t)$ denote the number of non-user and user firms, respectively.[9] We assume that for each k, $n_0(k,t)$ and $n_1(k,t)$, thus $n(k,t)$, are all publicly observed at the beginning of each period t. Note that it is possible that $n(k,t) = 0$ for some k, for instance, when $k^t_{os} > k$ and there is no non-user firm.

At the beginning of the second stage of period $t \geq 1$, after the open source use decisions have been made, the number of firms with technology level κ is denoted by $\eta(\kappa,t)$. At this stage, let $\eta_1(\kappa,t)$ and $\eta_0(\kappa,t)$ denote the number of user firms and non-user firms respectively, that is, $\eta(\kappa,t) = \eta_0(\kappa,t) + \eta_1(\kappa,t)$. Note that whenever $\kappa \neq \kappa_{os}$, $\eta_1(\kappa,t) = 0$.

At the beginning of the third stage of period $t \geq 1$, after the success/failure outcomes are realized, the number of firms that have technology level K is denoted by $N(K,t)$. Similarly, $N(K,t) = N_0(K,t) + N_1(K,t)$, where $N_0(K,t)$ and $N_1(K,t)$ denote the number of non-user and the number of user firms, respectively.

To sum up, in a given period t, the technology level of a firm is k^t_i at the beginning of the first stage, κ^t_i at the beginning of the second stage, and K^t_i at the beginning of the third stage, where $i = os$ denotes these technology levels for the open source technology. And, at any given period t, $n(k,t)$ is the number of firms with technology level k at the beginning of the first stage, $\eta(\kappa,t)$ is the number of firms with the technology level κ at the beginning of the second stage, and $N(K,t)$ is the number of firms with the technology level K at the beginning of the third stage. Note that, $\eta(\cdot,t) \geq n(\cdot,t)$, $n(\cdot,t+1) = N(\cdot,t)$ and $n_d(\cdot,t+1) = N_d(\cdot,t)$ for each $d \in \{0,1\}$ and $t \geq 1$.

3 Equilibrium Analysis: Finite Horizon

To solve this model, we study the Subgame Perfect Nash equilibrium of the 3 stage - T period game. We focus on symmetric equilibria, that is, we assume that in the third stage all firms with the same technology level pick the same quantity, in the second stage all firms with the same use decision and technology level pick the same success probability, and finally in the first stage all non-user firms with the same technology level make the same open source technology use decision.

3.1 Last Period: t=T

Before we start the equilibrium analysis with the last stage of the last period, we provide a lemma first, which deals with the first stage of the last period.

[9] Note that if $k < k_{os}$ or $k > k_{os}+1$, then $n_1(k,t) = 0$. When $k = k_{os}+1$, there may be user firms who have succeeded in innovation in the previous period.

Lemma 1 *In the last period, using the open source technology is a best response for any history of the game, for every firm, that is, $d_i^T = 1$ for all i.*

Proof. The proof is straightforward. Any firm with $k_i^T < k_{os}^T$ in the beginning of period T will be strictly better off using the open source technology since it will strictly lower its unit cost at no extra cost. Any firm with $k_i^T = k_{os}^T$ will be indifferent between using the open source technology and not using it. This is because, the unit cost will not depend on the use decision and since there is no future period, there is no future affect, thus all that matters is todays unit cost. Any firm with $k_i^T > k_{os}^T$ has no benefit from using the open source technology, since the technology level will not change with and without the open source, thus, the unit cost will not be affected. Again, since there is no future periods, there is no negative future effect of using the open source technology as well. Thus, such firms will also be indifferent between using and not using the open source technology. Thus, all firms at T will be weakly better off using the open source technology. ■

Lemma 1 says that in the very last period, it is optimal for every non-user firm to start using the open source technology. This is intuitive since there is no further period where using the open source technology would result in other firms free ride on the success of other firms. For firms with a technology level at least as high as the open source technology level, however, not adopting the open source technology is another best-response. We take the best response to be adopting the open source technology in this last period T. However, even if such firms choose not to adopt the open source technology (maybe due to an existing salvage value or maybe they just choose not adopting the open source technology if they are indifferent between adopting or not), the distribution of firms according to the technology levels will be the same as in the case where such firms choose to adopt the open source technology. An adopting firm will achieve a technology level that is the maximum of the current level and the level of the open source. Thus, for a firm with a technology level at least as high as the open source technology level, its technology level does not depend on its decision to adopt the open source. Since the unit cost of a firm depends only on its technology level, for such a firm, the unit cost is not affected by its decision to adopt the open source. Thus the distribution of firms according to their unit costs will not be affected even if these firms do not adopt the open source technology in the last period. Thus, it does not affect the rest of the analysis for the first $T-1$ periods.

Now we turn to the last stage of the last period. We denote the equilibrium number of firms of a certain technology level and thus a firm's expectation of the equilibrium number of firms of that technology level with the boldface counterparts. For instance, $\mathbf{N}(\mathbf{K}, \mathbf{t})$ denotes the equilibrium number of firms with technology level K at the beginning of the third stage of period t and also the expected number of such firms.

3.1.1 Third Stage: Cournot Competition

For each period t, a firm i observes own unit cost K_i^t but does not observe the innovation outcome of the other firms. Thus, each firm has an expectation of the technology distribution in the market, $\{\mathbf{N}(\mathbf{K},\mathbf{t})\}_K$. The expected inverse demand with K_i is $\mathbf{P}_{K_i^t} \equiv P(\mathbf{Q}_{K_i^t}) = A - \mathbf{Q}_{K_i^t}$. The expected total quantity can be decomposed; $\mathbf{Q}_{K_i^t} = q_{K_i^t} + \mathbf{Q}_{-i}^t$ where $\mathbf{Q}_{-i}^t = (\mathbf{N}(\mathbf{K_i^t},\mathbf{T})-1)q_{K_i^t} + \sum_{K \neq K_i^t} \mathbf{N}(\mathbf{K},\mathbf{T})q_{K^t}$. When a firm i with K_i^t chooses its quantity $q_{K_i}^t$, it solves the following problem

$$\max_{q_{K_i}^t} E[\pi_{K_i}^t] = (\mathbf{P}_{K_i^t} - c(K_i^t))q_{K_i}^t$$

Below, we characterize the Cournot equilibrium quantities and the expected Cournot profits for any period t. Let K and $\hat{K}$ be any two technology levels.

Lemma 2 *The Cournot equilibrium quantities must satisfy $q_K - q_{K'} = c(\hat{K}) - c(K)$ for any $K, \hat{K}$ in each t. The expected Cornout equilibrium profits are $E[\pi_{K_i}^t] = (q_{K_i^t})^2$, where*

$$q_{K_i^t} = \frac{1}{M+1}\left[A - c(K_i^t)\left(1 + \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}},\mathbf{t})\right) + \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}},\mathbf{t})c(\hat{K})\right]$$

Proof. The proof is in the Appendix, in Section 7.1. ■

The equilibrium Cournot quantity levels depend on the expected distribution of firms according to their technology levels, $\mathbf{N}(\hat{\mathbf{K}},\mathbf{t})$. If we write the quantity, $q_{K_i^t}$, in Lemma 2 as

$$q_{K_i^t} = \frac{1}{M+1}\left[A - c(K_i^t) - \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}},\mathbf{t})\left[c(K_i^t) - c(\hat{K})\right]\right]$$

one can see that, for a firm i with technology level K_i^t, the larger the expected number of firms with a higher technology level $\hat{K}$ (higher than K_i^t), the lower firm i's equilibrium quantity, since in that case $c(K_i^t) - c(\hat{K}) > 0$. This is intuitive since if the more advanced firms are expected to be larger in number, it will be harder to compete with them and thus quantity will be smaller. More importantly, if $K > \hat{K}$, then $q_K > q_{\hat{K}}$, since $c(\hat{K}) - c(K) > 0$, as $c(\cdot)$ is a decreasing function. Since the expected

Cournot profit levels at this stage are $E[\pi^t_{K_i}] = (q_{K_i^t})^2$ for each $K_i^t \geq K_{os}^t$, the more advanced a firm is at the technology ladder, the more it produces and the higher expected Cournot profit it gets. This is intuitive because the more advanced firms have lower unit costs, thus they have cost advantage in the Cournot competition. Thus, they produce more in the equilibrium and end up with higher profits in the Cournot competition.

In the last period T, by Lemma 1, there is no firm with $K_i^T < K_{os}^T$ after innovation realizations. Thus, in the equilibrium, it must be that $K_i^T \geq K_{os}^T$ for all i. Recall that in the first period, $k_{os}^1 = 1$ and there are firms with $k = 0$, $k = 1$ and $k = 2$ in this first period. Thus, it is possible that a non-user firm in the first period with a technology level $k = 2$ succeeds in every period, and when such a firm arrives at period T, it's technology level becomes $T + 2$. Thus, the technology level of firms in the last period may range between K_{os}^T and $T + 2$.[10] Thus, we have $K_i^T \in \{K_{os}^T, K_{os}^T + 1, ..., T + 2\}$ for each i.

3.1.2 Second Stage: Investment in Innovation

In the second stage, firms decide their investment levels by picking probability of success to advance a level in the technology level, that is, to decrease the unit cost. Each firm knows own unit cost at the beginning of this stage. Firms, after observing $n(k, t)$ for each k, do not observe the open source technology use decisions of other firms, d_j^t. They, however, know the expected number of firms for each k, $\eta(\mathbf{k}, \mathbf{t})$. Like the third stage, in this innovation stage, the technology level a firm has is at least $\kappa_{os}^T = k_{os}^T$ by Lemma 1. Thus, $\kappa_i^T \in \{k_{os}^T, k_{os}^T + 1, ..., T + 1\}$. Firm i with a technology level k_i^T and open source technology use decision d_i^T picks a probability of success of innovation, $p(d_i^T, k_i^T) \in (0, 1)$, by maximizing its expected profit. That is,

$$\max_{p} \quad pE[\pi^T_{\kappa_i+1}] + (1 - p)E[\pi^T_{\kappa_i}] - C(p)$$

Note that by Lemma 1, $d_i^T = 1$ for each i. Thus,

$$\kappa_i^T(d_i^T, k_i^T) = d_i^T \max\{k_i^T, k_{os}^T\} + (1 - d_i^T)k_i^T = \max\{k_i^T, k_{os}^T\}$$

is firm i's (potentially) new technology level. The equilibrium probability of firm i with k_i^T and $d_i^T = 1$ is obtained from the first order condition of the objective function in the above maximization problem.

[10] Note that $K_{os}^T \leq T + 2$, since even if open source community succeeds (with a firm with $k = 2$ joining the community in the first period) every period, the maximum technology level it can achieve is $T + 2$, as there are only T periods.

Then, we get

$$C'(p(d_i^T, k_i^T)) = E[\pi_{\kappa_i+1}^T] - E[\pi_{\kappa_i}^T] = q^2_{\kappa_i^T+1} - q^2_{\kappa_i^T} \tag{1}$$

The optimal success probability (investment level) is costly and it brings some expected benefit as well, thus, to get the optimal success probability, one must solve this trade-off: The optimal success probability should be such that its marginal cost must equal its marginal benefit. Equation 1 summarizes this condition, as the left hand side is the marginal cost of the success probability, and the right hand side is its expected benefit (the marginal expected profit of a one step higher technology level).

Here we provide a specification for the cost function, which we use to get a closed form for the condition in Equation 1. The specification we use here is $c(k_i^t) = \frac{1}{k_i^t+1}$ with $c(k_{os}^t) = \frac{1}{k_{os}^t+1}$. This specification does not affect any of the results we provide, but it is useful to see the relevant closed form of the condition above.[11] When using this specification, this condition boils down to

$$C'(p(d_i^T, k_i^T)) = (2q_{K_{os}^T} + \frac{2}{K_{os}^T+1} - \frac{1}{\kappa_i^T+2} - \frac{1}{\kappa_i^T+1})(-\frac{1}{\kappa_i^T+2} + \frac{1}{\kappa_i^T+1})$$

When $d_i^T = 1$ (by Lemma 1), we have $\kappa_i^T = \max\{k_{os}^T, k_i^T\}$. Also, $k_{os}^T = K_{os}^T$ since the innovation success by a user firm is not reflected until next period (in this case, it is never reflected because T is the last period). Then, for a firm with $k_i^T \leq k_{os}^T$, we get,

$$\begin{aligned} C'(p(1, k_i^T)) &= (2q_{K_{os}^T} + \frac{2}{K_{os}^T+1} - \frac{1}{K_{os}^T+2} - \frac{1}{K_{os}^T+1})(-\frac{1}{K_{os}^T+2} + \frac{1}{K_{os}^T+1}) \\ &= [2q_{K_{os}^T} + \frac{1}{(K_{os}^T+1)(K_{os}^T+2)}]\frac{1}{(K_{os}^T+1)(K_{os}^T+2)} \end{aligned}$$

For a firm with $k_i^T \leq k_{os}^T$, the marginal expected profit from success is affected adversely by the open source technology level, K_{os}^T, which is intuitive since the higher the level K_{os}^T, the lower the motivation to invest in success probability, because such a firm will be adopting this open source technology level anyways by Lemma 1.

[11] An alternative specification, for instance, is the one used in Modica (2012): when there is innovation the new unit cost is $c_{new} = (1-b)c_0$, where $b \in (0,1)$ and c_0 being the initial unit cost. Also, Aghion, Harris, Howitt and Vickers (2001), looking at the effect of production market competition and imitation on growth, assume that a firm's unit cost depends on its technology level and when a firm advances its technology level by one step, its unit cost decreases by some factor. However, since our model is dynamic, we want to have a tractable cost reduction process as a function of the technology level. With $c_{new} = (1-b)c$ type of reduction process, we would need to have $c(k) = (1-b)c(k-1) = (1-b)^2c(k-2) = (1-b)^kc_0$, which is not analytically tractable in our model.

3.1.3 First Stage: Using Open Source Technology

The first stage in period T is already discussed in the beginning of this section. And the result is summarized in Lemma 1: each firm (weakly) prefers to use the open source technology in the last period.

3.2 Next to Last Period: t=T-1

3.2.1 Third Stage: Cournot Competition

Since a firm's quantity choice $q_{K_i^t}$ only affects period-specific profit $\pi_{K_i}^t$ for each period t, the third stage problem in $T-1$ is generically the same as in period T. Thus, the quantity choice in this next to last period, $T-1$, is also given by Lemma 2. For the details of which, please see the proof of Lemma 2 in the Appendix, Section 7.1.

3.2.2 Second Stage: Investment in Innovation

For firm i with a technology level k_i^{T-1} and open source technology use decision d_i^{T-1},

$$\kappa_i^{T-1}(d_i^{T-1}, k_i^{T-1}) = d_i^{T-1} \max(k_i^{T-1}, k_{os}^{T-1}) + (1 - d_i^{T-1})k_i^{T-1}$$

is its (potentially) new technology level. Such a firm picks a probability of success of innovation, $p(d_i^{T-1}, k_i^{T-1}) \in (0, 1)$, by maximizing its expected profit. That is,

$$\max_p \quad p\left[E[\pi_{\kappa_i+1}^{T-1}] + \delta W_S(d_i^{T-1}, \kappa_i^{T-1} + 1)\right] + (1-p)\left[E[\pi_{\kappa_i}^{T-1} + \delta W_F(d_i^{T-1}, \kappa_i^{T-1})\right] - C(p)$$

where $W_S(d_i^{T-1}, \kappa_i^{T-1} + 1)$ is the expected continuation payoff from period T on, when at $T-1$ the open source technology use decision is d_i^{T-1} and the technology level at the end of period $T-1$ is $\kappa_i^{T-1} + 1$, with a success in innovation in the investment stage. Similarly, $W_F(d_i^{T-1}, \kappa_i^{T-1})$ is the expected continuation payoff from period T on, when at $T-1$ the open source technology use decision is d_i^{T-1} and the technology level at the end of period $T-1$ is κ_i^{T-1}, with a failure in innovation in the investment stage. Note that in this investment stage, possible technology levels are $\kappa_i^{T-1} \in \{0, 1, ..., T\}$.

Similar to period T, the equilibrium investment probability of firm i with k_i^{T-1} and d_i^{T-1} is then given by

$$C'(p(d_i^{T-1}, k_i^{T-1})) = E[\pi_{\kappa_i+1}^{T-1}] - E[\pi_{\kappa_i}^{T-1}] + \delta \left[W_S(d_i^{T-1}, \kappa_i^{T-1}+1) - W_F(d_i^{T-1}, \kappa_i^{T-1})\right] \quad (2)$$

$$= q^2_{\kappa_i^{T-1}+1} - q^2_{\kappa_i^{T-1}} + \delta \left[W_S(d_i^{T-1}, \kappa_i^{T-1}+1) - W_F(d_i^{T-1}, \kappa_i^{T-1})\right]$$

As in period T, the condition for the success probability takes both the marginal cost and marginal benefit into account. The left hand side of Equation 2 is the marginal cost of success probability and the right hand side is its expected marginal benefit (in terms of Cournot profits) which takes today's change in Cournot profits (with a success outcome) into account, as well as the future expected marginal change in future Cournot profits (when there is a sucess today), which is the term following the discount factor, in brackets.

3.2.3 First Stage: Using Open Source Technology

In this stage, each firm decides whether to use the open source technology or not. The decision depends on the current own technology level k_i^{T-1}, the current open source technology level, k_{os}^{T-1}, and the distribution of non-user and user firms for each technology level k, $n_0(k, T-1)$ and $n_1(k, T-1)$. The main trade-off for a non-user firm with $k_i^{T-1} < k_{os}^{T-1}$ is as follows. Starting to use the open source in this period will be beneficial for the current period through lower unit cost thus higher Cournot profit. Also, in case of a failure, success of a user firm will carry the technology level of one step higher and the firm will be able to use it from next period on. However, a potential innovation by such a firm will decrease the unit cost of other firms in the next period, because of the structure imposed by the GPL, and that will provide the other firms with the same cost advantage in the Cournot competition in the current period.

Let $V(d_i^{T-1}, k_i^{T-1})$ denote the expected payoff of firm i from period $T-1$ on. That is,

$$\begin{aligned} V(d_i^{T-1}, k_i^{T-1}) &= p(d_i^{T-1}, k_i^{T-1}) \left[E[\pi_{\kappa_i+1}^{T-1}] + \delta W_S(d_i^{T-1}, \kappa_i^{T-1}+1)\right] \\ &+ (1 - p(d_i^{T-1}, k_i^{T-1})) \left[E[\pi_{\kappa_i}^{T-1}] + \delta W_F(d_i^{T-1}, \kappa_i^{T-1})\right] - C(p(d_i^{T-1}, k_i^{T-1})) \end{aligned}$$

We show that the optimal open source technology use decision for a firm with $k_i^{T-1} < k_{os}^{T-1}$ is to use the open source technology, and the optimal open source use decision for a firm with $k_i^{T-1} \geq k_{os}^{T-1}$ is not to use the open source technology. These are summarized, respectively, in the two propositions below.

Proposition 1 *In the next to last period, if the technology level of a firm is at least as high as the open source technology level, then not using the open source technology is optimal for this firm, that is, if* $k_i^{T-1} \geq k_{os}^{T-1}$*, then* $d_i^{T-1} = 0$.

Proof. See the Appendix, Section 7.1. ■

The intuition for behind Proposition 1 is that whenever a firm starts the game with the same unit cost of the open source technology, there is no direct benefit from using the open source technology. However, using the open source technology makes the firm obliged to share its potential first period innovation, with other firms, in the second period, who choose to use the open source technology in the second period. This removes any potential cost advantage the firm could have in the second period quantity setting game. Hence any such firm will avoid using the open source technology.

Our second result says that any firm that produces the good at a higher unit cost than the unit cost of the open source technology will choose to use the open source technology even though GPL makes the firm obliged to share its potential innovation in the first period with the firms in the second period.

Proposition 2 *In the next to last period, if the technology level of a firm is one step behind the open source technology level, then using the open source technology is optimal for this firm, that is, if* $k_i^{T-1} = k_{os}^{T-1} - 1$*, then* $d_i^{T-1} = 1$.

Proof. See the Appendix, Section 7.1. ■

The intuition for Proposition 2 is as follows. For a firm with a lower technology level (that is, a higher unit cost) than the open source technology level, using the open source technology directly improves the firm's production technology hence its expected profit in period $T-1$. However, using the open source technology now and succeeding in cost reduction in the current period will provide the other users who have not succeeded with the cost reduction in the next period. Not using the open source technology now, and succeeding both now and in the next period may result in a cost advantage in the next period if the open source community fails both now and later. Thus, there is a tradeoff. The direct effect dominates the negative effect of using the open source technology, since the negative effect is of second degree. Therefore, the incentive to use the open source technology dominates the incentive not to use it.

Corollary 1 *In the next to last period, if the technology level of a firm is lower than the open source technology level, then using the open source technology is optimal for this firm, that is, if* $k_i^{T-1} < k_{os}^{T-1}$*, then* $d_i^{T-1} = 1$.

Corollary 1 is a direct implication of Proposition 2. The reason is as follows. Compared to the case $k_i^{T-1} = k_{os}^{T-1} - 1$, the positive/immediate effect of using the open source technology is larger than the one for case $k_i^{T-1} < k_{os}^{T-1} - 1$. However, the negative/future effect is the same in both cases. Thus, any firm who has a technology level less than the open source technology level, will prefer to start using the open source technology.

3.3 Any period $t < T$

Now we argue that the Proposition 1 is valid for any period $t < T$.

Proposition 3 *In any period* $t < T$*, if the technology level of a firm is at least as high as the open source technology level, then not using the open source technology is optimal for this firm, that is, if* $k_i^t \geq k_{os}^t$*, then* $d_i^t = 0$ *for all* i.

Proof. The proof is similar to the proof of Proposition 1. ■

We also argue that both the Proposition 2 and Corollary 1 are valid for any period $t < T$. Thus we prove the following proposition.

Proposition 4 *In any period* $t < T$*, if the technology level of a firm is lower than the open source technology level, then using the open source technology is optimal for this firm, that is, if* $k_i^t < k_{os}^t$*, then* $d_i^t = 1$.

Proof. The proof is similar to the proof of Proposition 2. Since we know that in period $T - 1$ each firm with a lower technology level than the open source's technology level, will use the open source technology. Thus, at period $T - 1$ each firm will have at least open source technology level. Thus, at $T - 2$, the reasoning to use the open source technology will be similar to the one in the proof of Proposition 2. ■

An immediate implication of the proposition above is given below.

Corollary 2 *At each period* $t \leq T$*, after the open source technology use decisions are made, there is no firm left with a technology level lower than the open source technology level, that is, for all* $t \leq T$*, we have* $\eta(k, t) = 0$ *for all* $k < k_{os}^t$.

Proof. By Propositions 1-4 , we know that at any period after the open source technology use decisions are made, a firm has a technology level either same as the open source technology level or higher. This is because, if a firm is behind the open source technology level, it is optimal for this firm to start using the open source technology. ■

Thus, putting all these together we get the following result.

Proposition 5 *At the beginning of the first stage of any period* $1 \leq t < T$*, there is no firm with a technology level that is behind the open source technology level by more than one level, that is,* $k_i^t \geq k_{os}^t - 1$ *for all* i*. At any period* t*, if the technology level of a firm is at least as high as the open source technology level, then the firm does not use the open source technology, that is, if* $k_i^t \geq k_{os}^t$*, then* $d_i^t = 0$*. At any period* t*, if the technology level of a firm is lower than the open source technology level, then the firm uses the open source technology, that is, if* $k_i^t = k_{os}^t - 1$*, then* $d_i^t = 1$*.*

Proof. The first argument for $t > 1$ follows from Corollary 2, which implies that in any period after the open source technology use decisions are made, there is no firm who is behind the open source technology. Thus, at the end of the period, after success/failure outcomes are realized, a non-user firm may end up falling behind the open source technology by at most one level, when it fails and the open source technology succeeds. Therefore, at the end of any period, that is, at the beginning of any period $t > 1$, all firms have a technology level at least as high as one level less than the open source technology level, which is $k_{os}^t - 1$. Note that, for $t = 1$, it follows from our assumption that the initial technology level of the open source is $k_{os}^1 = 1$, and any firm can at worst be $k_i^1 = 0$. The second and the third arguments are summarizing Proposition 1-4. ■

Thus, each firm stays out of the open source community as long as it's not behind the open source technology level. A firm which has the same technology level as the open source technology level may fail while the open source community may succeed, thus next period the firm is one step behind the open source technology level. At that point the firm starts using the open source technology.

In terms of the evolution of the open source community, in the light of the above results, as long as the open source technology users are successful in innovation, the open source community will grow and sweep the non-users, and the set of non-user firms will shrink. As long as, the open source community is not always successful in innovation, and the non-user firms that are ahead of the open source technology level succeed, there will be a set of proprietary firms with a higher technology level.

4 Equilibrium Analysis: Infinite Horizon

We have assumed that the time horizon is finite, and thus we were able to find the Subgame Perfect Nash equilibrium through a backward induction idea, starting with the very last period, T. However, it is also important to consider the infinite horizon case and see if our results still hold. To extend the model into an infinite horizon model, we need to consider how the equilibrium is defined by a sequence of both technology level distribution profile $K^t = \left(k_1^t, \cdots, k_m^t\right)$, where k_j^t is a technology level at least one firm has at period t, and its corresponding distribution of firms $N^t = \left(N_{k_1}^t, \cdots, N_{k_m}^t\right)$ where $\sum_{k=k_1^t}^{k_m^t} N_k^t = M$ for each t.

Our results extend to the infinite horizon case, where a key intuition in this case is as follows. For any firm i, the period-specific profit $\pi_{k_i}^t$ is increasing in the number of competitors with an inferior technology to its own, and it is decreasing in the number of competitors with a superior (or same) technology level, that is, $\frac{\partial \pi_{k_i}^t}{\partial N_{k-}^t} > 0$ and $\frac{\partial \pi_{k_i}^t}{\partial N_{k+}^t} < 0$, where $k_- < k_i \leq k_+$. Suppose that all other firms follow a strategy, $d_{-i}^t = 0$ when $k_{-i}^t \geq k_{os}^t$, and $d_{-i}^t = 1$ when $k_{-i}^t < k_{os}^t$. Then, for a firm with $k_i^t \geq k_{os}^t$, $d_i^t = 1$ may raise $N_{k_i}^{t'}$ while it lowers $N_{k-}^{t'}$ for some $t' > t$ when this firm i is successful in the investment stage. Also, using and not using the open source technology level will not affect the current technology level of this firm. Thus, this firm is better off by *not using* the open source technology. Similarly, for a firm with $k_i^t < k_{os}^t$, using the open source technology level hardly raises $N_{k_{i+}}^{t'}$ for any $t' > t$, and not using it does not change the distribution. However, by using it this firm jumps up in the technology ladder at no direct cost. Thus, such a firm is better off by *using* the open source technology. Under a mild condition, we prove the following result below.

Proposition 6 *In the infinite horizon case,*

(i) *if the technology level of a firm is higher than the open source technology level, then not using the open source technology is optimal for this firm, that is, if* $k_i^t > k_{os}^t$, *then* $d_i^t = 0$ *for all* i, *and*

(ii) *if the technology level of a firm is lower than the open source technology level, then using the open source technology is optimal for this firm, that is, if* $k_i^t < k_{os}^t$, *then* $d_i^t = 1$ *for all* i, *if either* δ *is small enough,* $\delta < \bar{\delta}$, *or the initial number of open source users is large enough,* $n(k_{os}, 1) > \bar{n}$, *or both, where* $\bar{\delta} \in (0, 1)$ *and* $0 < \bar{n} < M$.

Proof. See the Appendix (Section 7.2) for the proof and for the detailed analysis of the infinite horizon case. ■

While the non-user firm does not need to share its successes with other open source firms, the user firm has to share them with other user firms with one period lag, which may worsen the distribution of the technology levels against the favor of this firm. However, the continuation profit is discounted by δ, and if δ is small enough, this effect will be smaller relative to the positive effect due to the immediate profit difference. The finite horizon case in previous section reflects this point. Alternatively, if the current size of the open source community, $n(k_{os}, t)$, is large enough, the likelihood of the open source technology to advance in the technology will more likely and this will decrease the negative effect of the obligation to share the success since the open source technology will advance regardless. Thus, again the negative effect will be smaller relative to the positive effect of the larger immediate profit. Note that, if the initial number of open source firms, $n(k_{os}, 1)$ is large enough, then in the equilibrium $n(k_{os}, t)$ will also be large enough. Since $n(k_{os}, 1)$ is a model parameter, but $n(k_{os}, t)$ is an equilibrium object, we impose the restriction on $n(k_{os}, 1)$.

5 Discussion and Extensions

In this section, we discuss several relevant points and possible extensions.

Initial cost distribution: The initial unit cost distribution we have assumed is a specific one; k_i is equal to either 0, 1 or 2, where the open source technology level is at a level $k_{os} = 1$, to allow to have firms both at a lower and a higher technology levels relative to the open source technology level. Instead, we can assume a more general distribution. The initial technology levels of the firms can be $k \in \{0, 1, \cdots, K\}$ and $k_{os} \in \{0, 1, \cdots, K-1\}$, where $k_{os} \neq K$ ensures that there are firms with an initial technology level higher than the open source's technology level. Under this generalization, we believe that the idea behind our results will still be valid, that is, the firms that have a higher unit cost (lower technology level) than the source will use the open source, the firms that have the same or lower unit cost will choose not to use the source.

Leaving the open source community: Also, we assumed that the firms who start using the open source technology they do not leave the open source community. However, one could relax this assumption and let the users of open source technology leave it whenever they want. Under this specification, if, in the symmetric equilibrium, at any period if the user firms choose to stay in the open source community, then the outcome is equivalent to what we have provided in our equilibrium analysis above. However, if the user firms decide to leave the open source community the following period, then the number of user firms will be less relative to the specification in our model. Thus, the open source technology will evolve slower, in expectations, relatively. But, we believe that the use/not use decisions of the behind and ahead firms will not be affected.

Production cost: We have used a general unit cost function, which is decreasing in the technology level, and in the limit as the technology level goes to infinity, the unit cost goes to zero. With any such unit cost function, our results, Propositions 1-5, together with the lemmas and the corollaries are all valid, as we did not use any cost specification in the proofs of any of these results. However, we also employed a specific cost function, $c(k) = \frac{1}{k+1}$, just to see the closed forms of Cournot quantities and profits.

Continuous technology levels: When the technology levels are continuous, a firm with a private technology level slightly below the open source technology level, may prefer not to use the open source and stay private. However, if the technology level of the private firm is below a threshold, or say, if it is sufficiently lower than the open source technology level, then it will still be optimal for this firm to start using the open source technology, just like a firm, which is one level behind the open source technology, in our model with ladder-type technologies. That is, if $k_i < k_{os} - \overline{k}$, for some $\overline{k} > 0$, then firm i uses the open source. Thus, even with continuous technology level, firms with relatively lower technology levels will keep joining the open source community. And among private firms with technology level lower than that of the open source, only those with slightly below the open source level will stay private. So at any period t, as in our Proposition 5, which says "at the beginning of the first stage of any period $1 \leq t < T$, there is no firm with a technology level that is behind the open source technology level by more than one level", there will be no private firm that has a technology level that is considerably lower than the open source level. Also, over time, as the open source community grows, there will be more open source firms investing in innovation, relative to those private firms that are just slightly below the open source level, thus, these private firms may, at some point, fall behind the open source level more than $\overline{k}$ and they may end up using the open source next period. Thus, we believe that our prediction that the open source will grow and number of private firms will shrink overtime will still be the case with continuous technology levels, and we also believe that solving this threshold, $\overline{k}$ will not give us qualitatively a different result.

Adoption cost: Assumption 1 essentially says that there is no adoption cost to acquire the open source technology, which is new to the firm. However, there may be compatibility issues, cost of installing the new software and necessary training for using the new technology. Thus this adoption cost may not be zero. In this paper, we kept zero adoption cost assumption as a simplifying assumption and we leave it as an extension to see what happens when there is some positive adoption cost to acquire the new open source technology. We guess that the higher the adoption cost of the open source technology, the lower the incentives to join the open source community. However, the forces that derive our main result will still be there, and thus as far as the evolution of the open source community is concerned, the open source community will still grow, but at a smaller rate relative to the case with no adoption cost.

6 Conclusion

In this paper, we analyzed a simple dynamic model of innovation in cost reduction with an open source production technology present for the firms to freely use. We assumed, in the spirit of the GPL, that whenever a user succeeds in cost reduction innovation, it has to share this new technology with other users, in the subsequent periods. Because of this aspect of the GPL, we used a dynamic model, where in each period the firms decide whether to use the open source technology or not, invest in innovation, and engage in quantity competition. We characterized the optimal open source technology use decision of a firm as a function of its technology level relative to the open source technology level. A firm that has the same (or higher) technology as the open source finds it optimal not to use the open source technology. A firm that has a lower production technology level finds it optimal to use the source. These results show what principal effects of open source license on incentives for innovation and usage of the open source technology are. We believe that our model can be used as a tractable tool for further studies in open innovation.

7 Appendix

7.1 Finite Horizon

Proof of Lemma 2. Recall that the expected inverse demand with K_i is $\mathbf{P}_{K_i^t} \equiv P(\mathbf{Q}_{K_i^t}) = A - \mathbf{Q}_{K_i^t}$. The expected total quantity can be decomposed; $\mathbf{Q}_{K_i^t} = q_{K_i^t} + \mathbf{Q}_{-i}^t$ where $\mathbf{Q}_{-i}^t = \sum_{K \neq K_i^t} \mathbf{N}(\mathbf{K}, \mathbf{t}) q_{K^t} + (\mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) - 1) q_{K_i^t}$. Under Cournot competition, a firm i with K_i^t solves the following problem

$$\max_q E[\pi_{K_i}^t] = (\mathbf{P}_{K_i^t} - c(K_i^t))q$$

The first order condition is

$$\begin{aligned}\frac{\partial E[\pi_{K_i^t}]}{\partial q} &= P(\mathbf{Q}_{K_i^t}) - c(K_i^t) + \frac{\partial P(\mathbf{Q}_{K_i^t})}{\partial q} q \\ &= A - \mathbf{Q}_{K_i^t} - c(K_i^t) - q \\ &= A - c(K_i^t) - \mathbf{Q}_{-i}^t - 2q = 0\end{aligned}$$

Using symmetry, we get the following equilibrium condition;

$$2q_{K_i^t} = A - c(K_i^t) - \left(\sum_{K \neq K_i^t} \mathbf{N}(\mathbf{K}, \mathbf{t}) q_K + (\mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) - 1) q_{K_i^t} \right)$$

That is,

$$(\mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) + 1) q_{K_i^t} = A - c(K_i^t) - \sum_{K \neq K_i^t} \mathbf{N}(\mathbf{K}, \mathbf{t}) q_K$$

To see $q_K - q_{\hat{K}} = c(\hat{K}) - c(K)$ for any $K, \hat{K}$, first note that

$$(\mathbf{N}(\mathbf{K}, \mathbf{t}) + 1) q_K = A - c(K) - \sum_{k \neq K} \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k = A - c(K) - \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) q_{\hat{K}} - \sum_{k \neq K, \hat{K}} \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k$$

and similarly,

$$(\mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) + 1) q_{\hat{K}} = A - c(\hat{K}) - \sum_{k \neq \hat{K}} \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k = A - c(\hat{K}) - \mathbf{N}(\mathbf{K}, \mathbf{t}) q_K - \sum_{k \neq K, \hat{K}} \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k$$

Subtracting the two equations we get,

$$(\mathbf{N}(\mathbf{K}, \mathbf{t}) + 1) q_K - (\mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) + 1) q_{\hat{K}} = c(\hat{K}) - c(K) + \mathbf{N}(\mathbf{K}, \mathbf{t}) q_K - \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) q_{\hat{K}}$$

Thus, we get, $q_K - q_{\hat{K}} = c(\hat{K}) - c(K)$, that is, for any $K, \hat{K}$, we have $q_{\hat{K}} = q_K + c(K) - c(\hat{K})$. Now, substituting $q_{\hat{K}} = q_{K_i^t} + c(K_i^t) - c(\hat{K})$ into

$$\left(\mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) + 1\right) q_{K_i^t} = A - c(K_i^t) - \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) q_{\hat{K}} \tag{3}$$

we get

$$q_{K_i^t} = \frac{1}{\mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) + \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) + 1} \left[A - c(K_i^t) - \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) \left[c(K_i^t) - c(\hat{K}) \right] \right]$$

Since $\mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) + \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) = M$, we have

$$\begin{aligned} q_{K_i^t} &= \frac{1}{M+1} \left[A - c(K_i^t) - \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) \left[c(K_i^t) - c(\hat{K}) \right] \right] \\ &= \frac{1}{M+1} \left[A - c(K_i^t) \left(1 + \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) \right) + \sum_{\hat{K} \neq K_i^t} \mathbf{N}(\hat{\mathbf{K}}, \mathbf{t}) c(\hat{K}) \right] \end{aligned}$$

Note that this is a function of the technology distribution over firms at each period. Now, we calculate the equilibrium profit levels. First, note that by Equation 3, in the equilibrium, we have

$$q_{K_i^t} = A - c(K_i^t) - \sum_{K \neq K_i^t} \mathbf{N}(\mathbf{K}, \mathbf{t}) q_K - \mathbf{N}(\mathbf{K_i^t}, \mathbf{t}) q_{K_i^t} = A - c(K_i^t) - \sum_K \mathbf{N}(\mathbf{K}, \mathbf{t}) q_K$$

Using $\mathbf{Q}_K = \sum_k \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k$ and $\mathbf{P}_K = A - \mathbf{Q}_K = A - \sum_k \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k$, we get

$$E[\pi_{K_i}^t] = [\mathbf{P}_{K_i^t} - c(K_i^t)] q_{K_i^t} = [A - c(K_i^t) - \sum_k \mathbf{N}(\mathbf{k}, \mathbf{t}) q_k] q_{K_i^t} = (q_{K_i^t})^2$$

Proof of Proposition 1. We first show this result for $k_i^{T-1} = k_{os}^{T-1}$. Then, for $k_i^{T-1} > k_{os}^{T-1}$, it will be straightforward. If a firm with $k_i^{T-1} = k_{os}^{T-1}$ chooses not to use the open source technology, at $T-1$, then the firm will have the following expected payoff.

$$\begin{aligned} V(0, k_{os}^{T-1}) &= p(0, k_{os}^{T-1}) \left[E[\pi_{k_{os}+1}^{T-1}] + \delta W_S(0, k_{os}^{T-1} + 1) \right] \\ &+ (1 - p(0, k_{os}^{T-1})) \left[E[\pi_{k_{os}}^{T-1}] + \delta W_F(0, k_{os}^{T-1}) \right] - C(p(0, k_{os}^{T-1})) \end{aligned}$$

where $W_S(d^t, k^t)$ denotes the expected continuation payoff from period $t+1$ on, when at t the open source technology use decision is d^t and the technology level at the end of period t is k^t, with a success in innovation in the investment stage, and likewise, $W_F(d^t, k^t)$ denotes the expected continuation payoff from period $t+1$ on, when at t the open source technology use decision is d^t and the technology level at the end of period t is k^t, with a failure in innovation in the investment stage. Arranging this, we get

$$\begin{aligned} V(0,k_{os}^{T-1}) &= p(0,k_{os}^{T-1})\left[E[\pi_{k_{os}+1}^{T-1}] - E[\pi_{k_{os}}^{T-1}] + \delta[W_S(0,k_{os}^{T-1}+1) - W_F(0,k_{os}^{T-1})]\right] \\ &+ E[\pi_{k_{os}}^{T-1}] + \delta W_F(0,k_{os}^{T-1}) - C(p(0,k_{os}^{T-1})) \end{aligned}$$

Similarly, a firm with $k_i^{T-1} = k_{os}^{T-1}$ chooses to use the open source, at $T-1$, then the firm will have the following expected payoff.

$$\begin{aligned} V(1,k_{os}^{T-1}) &= p(1,k_{os}^{T-1})\left[E[\pi_{k_{os}+1}^{T-1}] - E[\pi_{k_{os}}^{T-1}] + \delta[W_S(1,k_{os}^{T-1}+1) - W_F(1,k_{os}^{T-1})]\right] \\ &+ E[\pi_{k_{os}}^{T-1}] + \delta W_F(1,k_{os}^{T-1}) - C(p(1,k_{os}^{T-1})) \end{aligned}$$

Recall that the optimal investment levels are

$$\begin{aligned} C'(p(0,k_{os}^{T-1})) &= E[\pi_{k_{os}+1}^{T-1}] - E[\pi_{k_{os}}^{T-1}] + \delta\left[W_S(0,k_{os}^{T-1}+1) - W_F(0,k_{os}^{T-1})\right] \\ C'(p(1,k_{os}^{T-1})) &= E[\pi_{k_{os}+1}^{T-1}] - E[\pi_{k_{os}}^{T-1}] + \delta\left[W_S(1,k_{os}^{T-1}+1) - W_F(1,k_{os}^{T-1})\right] \end{aligned}$$

Thus, we have

$$\begin{aligned} V(0,k_{os}^{T-1}) &= p(0,k_{os}^{T-1})C'(p(0,k_{os}^{T-1})) - C(p(0,k_{os}^{T-1})) + E[\pi_{k_{os}}^{T-1}] + \delta W_F(0,k_{os}^{T-1}) \\ V(1,k_{os}^{T-1}) &= p(1,k_{os}^{T-1})C'(p(1,k_{os}^{T-1})) - C(p(1,k_{os}^{T-1})) + E[\pi_{k_{os}}^{T-1}] + \delta W_F(1,k_{os}^{T-1}) \end{aligned}$$

Note also that $W_F(0,k_{os}^{T-1}) = W_F(1,k_{os}^{T-1})$. Then, we get

$$\begin{aligned} V(0,k_{os}^{T-1}) - V(1,k_{os}^{T-1}) &= p(0,k_{os}^{T-1})C'(p(0,k_{os}^{T-1})) - C(p(0,k_{os}^{T-1})) \\ &- [p(1,k_{os}^{T-1})C'(p(1,k_{os}^{T-1})) - C(p(1,k_{os}^{T-1}))] \end{aligned}$$

Note that, the function $F(p) = pC'(p) - C(p)$ is an increasing function. This is because, the first derivative is $F'(p) = C'(p) + pC''(p) - C'(p) = pC''(p) > 0$ since $C'' > 0$ and $p > 0$. Thus, if $p(0,k_{os}^{T-1}) > p(1,k_{os}^{T-1})$, then $V(0,k_{os}^{T-1}) > V(1,k_{os}^{T-1})$. Also, note that

$$C'(p(0,k_{os}^{T-1})) - C'(p(1,k_{os}^{T-1})) = \delta\left[W_S(0,k_{os}^{T-1}+1) - W_S(1,k_{os}^{T-1}+1)\right]$$

Now, we have $W_S(0, k_{os}^{T-1}+1) > W_S(1, k_{os}^{T-1}+1)$. To see this, note that in period T, if firm i has used the open source technology in period $T-1$, then it will be at the same technology level as the user firms by Lemma 1, because its success in period $T-1$ will be public for each user firm at T. Denote the expected payoff of this case by W^{eq}. That is, $W^{eq} = W_S(1, k_{os}^{T-1}+1)$. However, if firm i has not used the open source in period $T-1$, then its technology level will be the same as all the user firms with some probability ν and will be one step higher than all the user firms with probability $1-\nu$, where ν is the probability that at least one user firm is successful in period $T-1$. Denote the latter case's expected payoff with W^{sup}. Then, $W_S(0, k_{os}^{T-1}+1) > W_S(1, k_{os}^{T-1}+1)$ if $(1-\nu)W^{sup} + \nu W^{eq} > W^{eq}$. However, $W^{sup} > W^{eq}$ since in the former expected payoff firm i has a higher technology level than the latter. Thus, $W_S(0, k_{os}^{T-1}+1) > W_S(1, k_{os}^{T-1}+1)$, which in turn implies that $C'(p(0, k_{os}^{T-1})) > C'(p(1, k_{os}^{T-1}))$. Since $C'' > 0$, we get $p(0, k_{os}^{T-1}) > p(1, k_{os}^{T-1})$, which proves $V(0, k_{os}^{T-1}) > V(1, k_{os}^{T-1})$. When $k_i^{T-1} > k_{os}^{T-1}$, the proof works just the same. ■

Proof of Proposition 2. If a firm with $k_i^{T-1} = k_{os}^{T-1} - 1$ chooses not to use the open source, at $T-1$, then the firm will have the following expected payoff.

$$\begin{aligned} V(0, k_{os}^{T-1}-1) &= p(0, k_{os}^{T-1}-1)\left[E[\pi_{k_{os}}^{T-1}] + \delta W_S(0, k_{os}^{T-1})\right] \\ &+ (1-p(0, k_{os}^{T-1}-1))\left[E[\pi_{k_{os}-1}^{T-1}] + \delta W_F(0, k_{os}^{T-1}-1)\right] - C(p(0, k_{os}^{T-1}-1)) \end{aligned}$$

Arranging this, we get

$$\begin{aligned} V(0, k_{os}^{T-1}-1) &= p(0, k_{os}^{T-1}-1)\left[E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] + \delta W_S(0, k_{os}^{T-1}) - \delta W_F(0, k_{os}^{T-1}-1)\right] \\ &+ E[\pi_{k_{os}-1}^{T-1}] + \delta W_F(0, k_{os}^{T-1}-1) - C(p(0, k_{os}^{T-1}-1)) \end{aligned}$$

Recall the equilibrium investment probability of firm i with k_i^{T-1} and $d_i^{T-1} = 0$ is given by

$$C'(p(0, k_{os}^{T-1}-1)) = E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] + \left[\delta W_S(0, k_{os}^{T-1}) - \delta W_F(0, k_{os}^{T-1}-1)\right]$$

Plugging this into $V(0, k_{os}^{T-1}-1)$, we get

$$V(0, k_{os}^{T-1}-1) = p(0, k_{os}^{T-1}-1)C'(p(0, k_{os}^{T-1}-1)) - C(p(0, k_{os}^{T-1}-1)) + E[\pi_{k_{os}-1}^{T-1}] + \delta W_F(0, k_{os}^{T-1}-1)$$

$$V(1, k_{os}^{T-1}-1) = p(1, k_{os}^{T-1})C'(p(1, k_{os}^{T-1})) - C(p(1, k_{os}^{T-1})) + E[\pi_{k_{os}}^{T-1}] + \delta W_F(1, k_{os}^{T-1})$$

Note that $W_F(1, k_{os}^{T-1}) = W_F(0, k_{os}^{T-1} - 1)$, since in both cases the firm enters the second stage of the last period at the same technology level by Lemma 1, thus will have the same expected period T payoff. Thus, we get

$$\begin{aligned} V(1, k_{os}^{T-1} - 1) - V(0, k_{os}^{T-1} - 1) &= p(1, k_{os}^{T-1})C'(p(1, k_{os}^{T-1})) - C(p(1, k_{os}^{T-1})) \\ &- [p(0, k_{os}^{T-1} - 1)C'(p(0, k_{os}^{T-1} - 1)) - C(p(0, k_{os}^{T-1} - 1))] \\ &+ E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] \end{aligned}$$

Note that $E[\pi_{k_{os}-1}^{T-1}] = q_{k_{os}^{T-1}-1}^2$ and $E[\pi_{k_{os}}^{T-1}] = q_{k_{os}^{T-1}}^2$. Note also that $q_{k_{os}^{T-1}-1} < q_{k_{os}^{T-1}}$. Thus, $E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] > 0$. Now, if $p(1, k_{os}^{T-1}) > p(0, k_{os}^{T-1} - 1)$, then, since $pC'(p) - C(p)$ is an increasing function,

$$p(1, k_{os}^{T-1})C'(p(1, k_{os}^{T-1})) - C(p(1, k_{os}^{T-1})) > p(0, k_{os}^{T-1} - 1)C'(p(0, k_{os}^{T-1} - 1)) - C(p(0, k_{os}^{T-1} - 1)).$$

This, together with $E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] > 0$, implies that $V(1, k_{os}^{T-1} - 1) > V(0, k_{os}^{T-1} - 1)$. If, however, $p(1, k_{os}^{T-1}) < p(0, k_{os}^{T-1} - 1)$, then we have

$$\begin{aligned} V(1, k_{os}^{T-1} - 1) - V(0, k_{os}^{T-1} - 1) &= p(1, k_{os}^{T-1})C'(p(1, k_{os}^{T-1})) - C(p(1, k_{os}^{T-1})) \\ &- [p(0, k_{os}^{T-1} - 1)C'(p(0, k_{os}^{T-1} - 1)) - C(p(0, k_{os}^{T-1} - 1))] \\ &+ E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] \\ &= C(p(0, k_{os}^{T-1} - 1)) - C(p(1, k_{os}^{T-1})) \\ &+ p(1, k_{os}^{T-1})C'(p(1, k_{os}^{T-1})) \\ &- p(0, k_{os}^{T-1} - 1)C'(p(0, k_{os}^{T-1} - 1)) \\ &+ E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] \\ &> C(p(0, k_{os}^{T-1} - 1)) - C(p(1, k_{os}^{T-1})) \\ &- C'(p(0, k_{os}^{T-1} - 1)) + E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] \\ &= C(p(0, k_{os}^{T-1} - 1)) - C(p(1, k_{os}^{T-1})) \\ &- \left[E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] + \delta W_S(0, k_{os}^{T-1}) - \delta W_F(0, k_{os}^{T-1} - 1)\right] \\ &+ E[\pi_{k_{os}}^{T-1}] - E[\pi_{k_{os}-1}^{T-1}] \\ &= C(p(0, k_{os}^{T-1} - 1)) - C(p(1, k_{os}^{T-1})) \\ &- [\delta W_S(0, k_{os}^{T-1}) - \delta W_F(0, k_{os}^{T-1} - 1)] \end{aligned}$$

The inequality follows since $C' > 0$ and $p(0, k_{os}^{T-1} - 1)$ can be at most 1. Note that $C(p(0, k_{os}^{T-1} - 1)) - C(p(1, k_{os}^{T-1})) > 0$ since $C' > 0$. Also note that, $W_S(0, k_{os}^{T-1}) = W_F(0, k_{os}^{T-1} - 1)$. This is because at the end of the first stage of period T (after open source technology use decisions are made), the technology level of the firm will be the same under the both cases (again by Lemma 1), and since the firm is not using the open source technology in period $T-1$, its own success at period $T-1$ will not be available for the other firms in period T, thus the two expected payoffs are the same. Thus,

$$V(1, k_{os}^{T-1} - 1) - V(0, k_{os}^{T-1} - 1) > C(p(0, k_{os}^{T-1} - 1)) - C(p(1, k_{os}^{T-1})) > 0$$

which completes the proof. ■

7.2 Infinite Horizon

Here we layout the infinite horizon version of our model. When there is infinitely many periods, we need to conduct an analysis using the value function and use steady state dynamics. At the beginning of period t, a non-user firm i with technology level k_i^t, makes a decision d_i^t whether to use the open source or not. After d_i^t is realized, the technology level of firm i is then κ_i^t. Then in the second stage of the same period, investment in cost reducing (or technology level advancement) is made, p_i^t. After the success/ failure outcome is realized, the new technology level of firm i is K_i^t, which is also equal to k_i^{t+1}. Thus, the cycle is represented as $k_i^t \Rightarrow_{d_i^t} \kappa_i^t \Rightarrow_{p_i^t} K_i^t$, where $K_i^t = k_i^{t+1}$. Given K_i^t, in the last stage of period t, firm i picks its quantity level, $q_{K_i^t}$ in the Cournot competition.

At each period t, firm i chooses a triple of $\left(d_i^t, p_i^t, q_i^t\right)$ where each decision also depends on the firm's technology level right before the decision made, the distribution of technology levels of other firms, and the innovation probability depends also on the usage decision of the firm. More precisely, $d_i^t = d(k_i^t, N^t)$, $p_i^t = p(d_i^t, \kappa_i^t, N^t)$ and $q_i^t = q(K_i^t, N^t)$, where $N^t = \left(N_{k_1}^t, \cdots, N_{k_M}^t\right)$ with $\sum_{k^t=k_1^t}^{k_M^t} N_k^t = M$ for each t, with with N_k^t denoting the number of firms with technology level k at period t.[12] An equilibrium is characterized by $\left\{d_i^t, p_i^t, q_i^t\right\}_{i,t}$. The value function of a firm i at period t is

$$\begin{aligned} V(d_i^t, \kappa_i^t, N^t) &= p(d_i^t, \kappa_i^t, N^t)\left[E[\pi_{k_i+1}^t] + \delta V(d_i^{t+1}, \kappa_i^t + 1, N^{t+1})\right] \\ &+ (1 - p(d_i^t, \kappa_i^t, N^t))\left[E[\pi_{k_i}^t] + \delta V(d_i^{t+1}, \kappa_i^t, N^{t+1})\right] - C(p(d_i^t, \kappa_i^t, N^t)) \end{aligned}$$

[12] Here we abuse notation. Each N^t in the decision variables represents the distribution of firms of all technology levels at the relevant stage of the period. For instance, in $d(k_i^t, N^t)$, N^t refers to the distribution right before the use decisions are made, in $p(d_i^t, \kappa_i^t, N^t)$, N^t refers to the distribution after the use decisions are made, and in $q(K_i^t, N^t)$ it refers to the distribution after the success/failure outcomes are realized.

According to the open source license, $d_i^{t'} = 1$ if $d_i^{t=1}$ for any t, t' where $t' > t$. Thus, it is enough to know d_i^t to define the value function at period t, rather than $\mathbf{d}_i^t = (d_i^1, d_i^2, \cdots, d_i^t)$. Also note that the distribution of others' decisions $\mathbf{d}^t = (\mathbf{d}_1^t, \mathbf{d}_2^t, \cdots, \mathbf{d}_M^t)$ is already taken into account through N^t.

In Stage 3, firm i's period-specific profit maximization problem is $\max_{q_i^t} E[\pi_{k_i}^t] = (\mathbf{P}_{K^t} - c(k_i^t))q_i^t$. The optimal production level will be

$$q_{k_i}^t = \frac{1}{M+1}\left[A - c(k_i) - \sum_{k_{-i}} \mathbf{N}_{k_{-i}} \left(\mathbf{c}(k_i^t) - \mathbf{c}(k_{-i})\right)\right].$$

In Stage 2, the optimal investment level $p_i^t = p(d_i^t, \kappa_i^t, N^t)$ satisfies

$$C'(p(d_i^t, \kappa_i^t, N^t)) = E[\pi_{\kappa_i+1}^t] - E[\pi_{\kappa_i}^t] + \delta\left[V(d_i^{t+1}, \kappa_i^t + 1, N^{t+1}) - V(d_i^{t+1}, \kappa_i^t, N^{t+1})\right] \tag{4}$$

In Stage 1, the optimal use decision, $d_i^t = d(k_i^t, N^t)$, is determined by the comparison of $V(1, \kappa_i^t, N^t)$ and $V(0, \kappa_i^t, N^t)$. Arranging terms, we can write $V(d_i^t, \kappa_i^t, N^t)$ as

$$\begin{aligned} V(d_i^t, \kappa_i^t, N^t) &= p(d_i^t, \kappa_i^t, N^t)\left[E[\pi_{\kappa_i+1}^t] - E[\pi_{\kappa_i}^t] + \delta\left\{V(d_i^{t+1}, \kappa_i^t + 1, N^{t+1}) - V(d_i^{t+1}, \kappa_i^t, N^{t+1})\right\}\right] \\ &\quad + E[\pi_{\kappa_i}^t] + \delta V(d_i^{t+1}, \kappa_i^t, N^{t+1}) - C(p(d_i^t, \kappa_i^t, N^t)) \end{aligned}$$

Proof of Proposition 6. By using Equation 4, we can rewrite $V(d_i^t, \kappa_i^t, N^t)$ as

$$V(d_i^t, \kappa_i^t, N^t) = p(d_i^t, \kappa_i^t, N^t)C'(p(d_i^t, \kappa_i^t, N^t)) - C(p(d_i^t, \kappa_i^t, N^t)) + E[\pi_{k_i}^t] + \delta V(d_i^{t+1}, \kappa_i^t, N^{t+1})$$

The difference of value functions between *not-use* and *use*, simplifying the notation, is given by $V(0, \kappa_i^t) - V(1, \max\{\kappa_i^t, \kappa_{os}^t\})$. Denoting $\max\{\kappa_i^t, \kappa_{os}^t\} = \kappa_{max}^t$, we can write

$$\begin{aligned} V(0, \kappa_i^t) - V(1, \max\{\kappa_i^t, \kappa_{os}^t\}) &= V(0, \kappa_i^t) - V(1, \kappa_{max}^t) \\ &= p(0, \kappa_i^t)C'(p(0, \kappa_i^t)) - C(p(0, \kappa_i^t)) \\ &- \left[p(1, \kappa_{max}^t)C'(p(1, \kappa_{max}^t)) - C(p(1, \kappa_{max}^t))\right] + \delta(V(0, \kappa_i^t) - V(1, \kappa_{max}^t)) \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned}(1-\delta)\left[V(0,\kappa_i^t)-V(1,\kappa_{max}^t)\right] &= p(0,\kappa_i^t)C'(p(0,\kappa_i^t))-C(p(0,\kappa_i^t)) \\ &- \left[p(1,\kappa_{max}^t)C'(p(1,\kappa_{max}^t))-C(p(1,\kappa_{max}^t))\right]\end{aligned}$$

Note that the function $F(p) \equiv pC'(p) - C(p)$ is increasing in p, since its first derivative is given by $F'(p) = C'(p) + pC''(p) - C'(p) = pC''(p) > 0$ where $C'' > 0$ and $p > 0$. Thus, if $p(0,\kappa_i^t) > p(1,\kappa_{max}^t)$, then $V(0,\kappa_i^t) > V(1,\kappa_{max}^t)$, and it is optimal to not use the open source technology. If $p(0,\kappa_i^t) < p(1,\kappa_{max}^t)$, then $V(0,\kappa_i^t) < V(1,\kappa_{max}^t)$, and it is optimal to use the open source technology.

To see whether $p(0,\kappa_i^t) > p(1,\kappa_{max}^t)$ or $p(0,\kappa_i^t) < p(1,\kappa_{max}^t)$, we use Equation 4 again to get

$$C'(p(0,\kappa_i^t)) - C'(p(1,\kappa_{max}^t)) = \delta\left[V(0,\kappa_i^t+1) - V(0,\kappa_i^t) - \left(V(1,\kappa_{max}^t+1) - V(1,\kappa_{max}^t)\right)\right] \tag{5}$$

Also note that for any firm i, the period-specific profit $\pi_{k_i}^t$ is increasing in the number of competitors with an inferior technology to its own, and it is decreasing in the number of competitors with a superior (or same) technology level, that is, $\frac{\partial \pi_{k_i}^t}{\partial N_{k-}^t} > 0$ and $\frac{\partial \pi_{k_i}^t}{\partial N_{k+}^t} < 0$, where $k_- < k_i \leq k_+$.

(i) **A firm that is ahead of the open source technology level:** Now, suppose firm i is a non-user firm, at the end of $t-1$, with $k_i^t > k_{os}^t$, that is, i is a firm that is ahead of the open source technology level at the start of period t. For such a firm, using the open source does not change own technology level, $\kappa_{max}^t = \kappa_i^t$. Also, the marginal benefit of a jump in the technology for a non-user firm is larger than the one for a user firm, because the non-user firm will get the cost advantage on its own starting right after the jump in technology level, but if it starts using the open source technology, other open source firms will adopt this firm's technology level and the firm will need to share that jump with all these other user firms from next period on (due to GPL), that is, N_{k-}^t will decrease if the firm joins the open source community. Thus, the marginal benefit of the jump for this firm will be smaller than the one for the non-user firm. Thus, we have $V(0,\kappa_i^t+1) - V(0,\kappa_i^t) > V(1,\kappa_i^t+1) - V(1,\kappa_i^t)$. This implies that the right hand side of Equation 5 is positive, thus, $C'(p(0,\kappa_i^t)) > C'(p(1,\kappa_i^t))$. Since $C'' > 0$, we get $p(0,\kappa_i^t) > p(1,\kappa_i^t)$, implying $V(0,\kappa_i^t) > V(1,\kappa_i^t)$. This establishes that a non-user firm, with $\kappa_i^t > \kappa_{os}^t$, does not use the open source technology. Thus, for a firm the steady state decision, when it is ahead of the open source technology level, is $d^*(k) = 0$ for $k > k_{os}$. A firm that is ahead of the open source technology level in the first period will adopt this decision, and for any other period

$t > 1$, it will not use the open source technology level as long as it is ahead of it.

(ii) **A firm that is behind the open source technology level:** Now suppose, firm i is a non-user firm at the end of $t-1$, with $k_i^t < k_{os}^t$, that is, i is a firm that is behind the open source technology level at the start of period t. For such a firm, using the open source technology changes its technology level, $\kappa_{max}^t = \kappa_{os}^t$.[13] To show that using the open source technology level is optimal for this firm, first we make two observations.

First, if it is optimal for a firm with $k_i^t = k_{os}^t - 1$ to use the open source technology, then it is also optimal to use it for a firm with $k_i^t < k_{os}^t - 1$. This is clearly the case since when for the firm with $k_i^t = k_{os}^t - 1$, the net benefit of jumping one step up in the technology ladder dominates the net benefit of not using the open source, jumping more than one step must also dominate the net benefit of not using the open source. Thus, we can restrict attention to a firm that is behind by only one step: $k_i^t = k_{os}^t - 1$.

Secondly, if it optimal to use the open source technology for a firm with $k_i^t = k_{os}^t - 1$ when the distribution of firms' technology levels is $\{n(k_{os}-1,t), n(k_{os},t), n(k_{os}+1,t)\}$ with $M = n(k_{os}-1,t) + n(k_{os},t) + n(k_{os}+1,t)$, then it is also optimal for this firm to use the open source when the distribution of firms' technology levels is $\{n(k_{os}-1,t), n(k_{os},t), \hat{n}(k_{os}+1,t), \hat{n}(k_{os}+2,t), ..., \hat{n}(k_{os}+K,t)\}$ with $M = n(k_{os}-1,t) + n(k_{os},t) + \sum_{s=1}^{K} \hat{n}(k_{os}+s,t)$. The reason for this observation is that the benefit from using the open source under the former distribution is smaller than the benefit under the latter one. This is because the latter is a relatively worse distribution for this firm, and therefore a jump from $k_{os}^t - 1$ to k_{os}^t has a higher benefit under the latter than under the former. Thus, if it is optimal to use the open source under the former distribution, it must also be optimal under the latter distribution.

By these two above observations, it suffices to show that it is optimal to use the open source for a firm when it is behind the open source technology by one step and the distribution of firms' technology levels is given by $\{n(k_{os}-1,t), n(k_{os},t), n(k_{os}+1,t)\}$ with $M = n(k_{os}-1,t) + n(k_{os},t) + n(k_{os}+1,t)$.

Assuming that every other firm follows the usage decision rule, $d^*(k_j) = 1$ for $k_j < k_{os}$ and $d^*(k_j) = 0$ for $k_j > k_{os}$, we will show that the usage decision rule d^* is a best response for this firm. To see this, consider the possible success/fail realization of this firm in the period t, which is behind the open source technology by one step, at the first stage of period t.

First, consider the realization where the firm is successful in innovation. If the firm had chosen to

[13] Note that comparing $V(0, \kappa_i^t + 1) - V(0, \kappa_i^t)$ and $V(1, \kappa_{os}^t + 1) - V(1, \kappa_{os}^t)$ is not relevant since these two differences are evaluated at different technology levels, since $\kappa_{os}^t \neq \kappa_i^t$.

use the open source, the firm will have a technology level $k_{os}+1$ at the competition stage of the current period and the within period profit will be $\pi_{k_{os}+1}$. Let the continuation expected profit level be $V^{d=1}_{k_{os}+1}$. If this firm had chosen not to use the open source technology, then with the success in innovation its technology level will be k_{os} at the competition stage of the current period and the within period profit will be $\pi_{k_{os}}$. Let the continuation expected profit level be $V^{d=0}_{k_{os}}$.[14] Using the open source has an immediate marginal benefit: $\pi_{k_{os}+1} - \pi_{k_{os}}$. However, the effect of using the open source technology, in terms of the expected continuation profit can be negative: $V^{d=0}_{k_{os}}$ may be larger than $V^{d=1}_{k_{os}+1}$ even though the starting technology level is one step behind that of the latter. This is because, regarding the former continuation profit, the firm does not need to share its successes with other open source firms, but in the latter it has to share them with one period lag, which may worsen the distribution of the technology levels against the favor of this firm. However, the continuation profit is discounted by δ, and if δ is small enough, this effect will be smaller relative to the positive effect due to the immediate profit difference. Or, if the current size of the open source community, $n(k_{os}, t)$, is large enough (which will be the case if the initial size of the open source community, $n(k_{os}, 1)$, is large enough), the likelihood of the open source technology to advance in the technology will more likely and this will decrease the negative effect of the obligation to share the success since the open source technology will advance regardless. Thus, again the negative effect will be smaller relative to the positive effect of the larger immediate profit. Notice that as long as a non-user firm's innovation outcome at period t is made independently, we may ignore its effect on the expected technology distribution. Hence, if either one of these two above conditions is met, we will have $\pi_{k_{os}+1} - \pi_{k_{os}} > \delta[V^{d=0}_{k_{os}} - V^{d=1}_{k_{os}+1}]$, and it will be optimal to use the open source technology, when the firm is behind it.

Now, we look at the other realizations, that is, the firm's investment in innovation fails at the current period. Once again using the open source technology level now has an immediate positive effect: $\pi_{k_{os}} - \pi_{k_{os}-1}$. And, regarding the continuation profit, the effect of using the open source may be negative: $V^{d=0}_{k_{os}-1} - V^{d=1}_{k_{os}}$. However, since the firm has failed, this effect (potentially negative) is weaker than the one when the firm has succeeded (using the open source does not affect the distribution of technology levels in the next period, since the firm has no success to share). Thus, if either one of the conditions given above holds, then we will once again have $\pi_{k_{os}} - \pi_{k_{os}-1} > \delta[V^{d=0}_{k_{os}-1} - V^{d=1}_{k_{os}}]$, and it again will be optimal to use the open source technology, when the firm is behind it.

The optimal decision whether to use or not to use the open source technology is based on the overall

[14] Here we abuse notation, by dropping t and inserting the usage decision, d, as a superscript.

expected profit from using and the expected profit from not using. The net effect of using the open source technology on the overall expected profit is a linear combination of the effects depicted for the two success/fail realizations, weighted by the probabilities of innovation. Assuming at least one of the two conditions is met (either δ is small enough, or $n(k_{os}, 1)$ is large enough, or both), since using the open source technology makes the firm better off under each realization, the expected overall effect of using the open source technology will be positive relative to not using it. It follows that it is optimal for this firm i to use the open source technology, if it is behind it.

Therefore, the usage decision function will be $d^*(k_i) = 1$ for $k_i < k_{os}$ and $d^*(k_i) = 0$ for $k_i > k_{os}$ for any i. In the equilibrium, each firm will use this decision rule $d^*(k_i)$ in every period. ■

7.3 The definitions for all parameters and the decision variables:

Parameters	Definition
M	Number of firms.
T	Number of periods.
δ	Discount factor, common for every firm.
$c(k_i^1)$	Firm i's initial unit cost.
$c(k_{os}^1)$	Open source technology's initial unit cost.
k_i^1	Initial technology level of firm i.
k_{os}^1	Initial technology level of the open source technology.
$n(k, 1)$	Initial number of firms with technology level $k \in \{0, 1, 2\}$.

Table 1. Model Parameters

Decision Variables	Definition
d_i^t	Open source technology use decision of firm i at the first stage of period t.
$p(d_i^t, \kappa_i^t)$	Probability of success of firm i with a technology level κ_i^t and decision d_i^t.
$q_{K_i^t}$	Production level of firm i when it has a technology level K_i^t.

Table 2. Decision Variables

Flow Variables	Definition
$c(k_i^t)$	Firm i's unit cost in period $t > 1$.
$c(k_{os}^t)$	Open source technology's unit cost in period $t > 1$.
k_i^t	Technology level of firm i at the beginning of period $t > 1$.
k_{os}^t	Open source technology level at the beginning of period $t > 1$.
κ_i^t	Technology level of firm i, at the beginning of the second stage of period t.
κ_{os}^t	Open source technology level, at the beginning of the second stage of period t.
K_i^t	Technology level of firm i in period t, after the success/failure realization of the investment level $p(d_i^t, k_i^t)$.
K_{os}^t	Open source technology level in period t, after the success/failure realizations of the investment levels of all the open source technology users.
$n(k,t)$	Number of firms with technology level k at the beginning of the first stage of period t.
$n_1(k,t)$	Number of user firms with technology level k at the beginning of the first stage of period t.
$n_0(k,t)$	Number of non-user firms with technology level k at the beginning of the first stage of period t.
$\eta(\kappa,t)$	Number of firms with the technology level κ at the beginning of the second stage of period t.
$\eta_1(\kappa,t)$	Number of user firms with the technology level κ at the beginning of the second stage of period t.
$\eta_0(\kappa,t)$	Number of non-user firms with the technology level κ at the beginning of the second stage of period t.
$N(K,t)$	Number of firms with the technology level K at the beginning of the third stage of period t.
$N_1(K,t)$	Number of user firms with the technology level K at the beginning of the third stage of period t.
$N_0(K,t)$	Number of non-user firms with the technology level K at the beginning of the third stage of period t.
$\mathbf{N}(\mathbf{K},\mathbf{T})$	Expected number of firms with technology level K at the beginning of the third stage of period t.

$\mathbf{Q}_{-i}^t$	Expected total quantity of all firms except firm i at the end of Cournot competition in period t.
$\mathbf{Q}_{K_i^t}$	Expected total quantity demanded a firm with K_i^t expects at the end of Cournot competition in period t.
$\pi_{K_i}^t$	Firm i's realized profit level at the end of Cournot competition in period t.
$W_S(d_i^{T-1}, \kappa_i^{T-1}+1)$	Expected continuation payoff from period T on, when at $T-1$ the open source technology use decision is d_i^{T-1} and the technology level at the end of period $T-1$ is $\kappa_i^{T-1}+1$, with a success in innovation in the investment stage.
$W_F(d_i^{T-1}, \kappa_i^{T-1})$	Expected continuation payoff from period T on, when at $T-1$ the open source technology use decision is d_i^{T-1} and the technology level at the end of period $T-1$ is κ_i^{T-1}, with a failure in innovation in the investment stage.

Table 3. Flow Variables

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