A Dynamic Investment Model under Time-Inconsistency

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Abstract

In this paper, we analyze a three period dynamic investment model where a principal interacts with an agent who is time-inconsistent. We use $\beta\delta$-preferences to capture time-inconsistency as it is widely used in the literature. Every period, the agent invests an amount and then the principal, observing the investment amount, makes an offer or not, and then the agent accepts the offer or not. For any δ and β combination, we find the optimal investment streams for both sophisticated and naive type of time-inconsistent agent. For a sophisticated type of agent, for βs above a threshold, we find that for intermediate values of δ, as the degree of time-inconsistency increases, the game ends in the second period rather than the first period, and for relatively high values of δ, as the degree of time-inconsistency increases, the game ends in the second period rather than the third period. When the agent is naive, however, for a range of δ values, below a certain β value, the agent (after realizing that he is time-inconsistent) changes his mind about his investment decision and ends the game in the second period, even though he made an investment in the first period with which he intended to end the game in the third period. We also find that for all δ and β values naive agent (erroneously) makes an overall investment that is at least as big as that of the sophisticated agent. Also, for the principal, it is better that the agent is naive rather than sophisticated for all δ and β values.

Keywords :dynamic investment, time-inconsistency, $\beta\delta$-preferences, sophisticated agent, naive agent. *JEL* **Classification Numbers:** D03, D82, D86.

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1 Introduction

Experiments and daily observations provide supporting evidence that people show present biased preferences, or more specifically time-inconsistent behavior when making decisions over a period of time. In other words, people's preferences change over time, that is, something that is desirable at a point of time will not be that desirable at a different point in time. Thus, people have a tendency to delay costly actions, for instance, starting up writing a thesis and going to the gym. However, they want to receive instant gratification, yielding time-inconsistent behavior.[1]

In this paper, we focus on investment decisions of an agent who is interacting with a principal over time. As a motivating example, we think of various education levels one can pursue, and at a certain level one starts working, potentially with higher wages and benefits when the investment in education is higher.[2] We directly see people getting different levels of education, namely, some people only get a bachelors degree, some get a master of arts degree, and some pursue a doctoral degree. In the light of this example, we ask the following questions: Why do people end up with different investment levels, and what are the differences in people's preferences which cause this variance in investment levels? Of course, in education example, there are many reasons and explanations (for instance job market signaling is one of them[3]). What we want to do is to see if there is a link between the time-inconsistency of an agent and the different investment levels he pursues. More specifically, can we explain different investment levels through different time-inconsistent type of preferences? Can we match time-consistent and time-inconsistent (sophisticated or naive) preferences with different investment levels?

In the model we consider, there is an agent and a principal interacting over three periods. Each period, the agent picks an investment level $\theta_t \in [\theta_{t-1}, \infty)$ at cost $c(\theta_t - \theta_{t-1})$. The principal observes θ_t and then offers the agent a partnership or not. Then, the agent accepts or rejects. If the agent accepts, the production takes place and the total revenue, $R(\theta_t)$, is realized. Then, they share the total revenue generated, where the agent gets $sR(\theta_t)$ and the principal gets the rest. If the agent rejects, then they move on to the next period. If there is no accepted offer at

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[1]See Frederick, Loewenstein, O'Donoghue (2002), for an extensive overview of the literature, and see Loewenstein and Prelec (1992) for an extensive survey on anomalies in intertemporal choice.

[2]In Harris and Holmstorm (1982), they show that in the equilibrium wages never decrease with age and increase only when the worker's market value increases above his current wage.

[3]See Spence (1973).

the end of the third period, the game ends and they get no revenue and the agent incurs the cost of all investment he made. We assume that the principal is aware of the type of the agent and provide results within this framework, but we also discuss what happens when the principal does not have that piece of information.

We assume that the agent has time-inconsistent preferences. Time-inconsistent preferences are often captured by hyperbolic discount factors, that is, a discounting scheme that reflects the change in the preferences over time. More specifically, the agent's time-inconsistency is modeled by $\beta\delta$-preferences.[4] An agent with $\beta\delta$-preferences has a discount factor $\beta\delta$ between the current and the next period, and δ between any other pair of successive periods. In other words, the agent's preference for a payoff in date t over a payoff in date $t+1$ is stronger as date t gets closer. A time-inconsistent agent may or may not be aware of his inconsistency: A sophisticated agent is fully aware of his inconsistency in the sense that he correctly predicts how his future selves will behave. A naive agent, on the other hand, is not aware of his inconsistency, in the sense that he mispredicts the behavior of his future selves through an overestimated β.

The main results of our paper are as follows.[5] Through backward induction, we find the optimal investment streams for both sophisticated and naive type of time-inconsistent agent. For a sophisticated type of agent, we find that for high values of β, as δ increases, the game ends at a higher period, including all $t = 1, 2, 3$. On the other hand, for relatively low values of β, the game ends in the first period for low δ values and ends in the third period for high δ values, and never ends in the second period. When the agent is naive, however, for a range of δ values, below a certain β value, the agent (after realizing that he is time-inconsistent) changes his mind about his investment decision and finalizes the game in the second period, even though he made an investment in the first period with which he intended to end the game in the third period. We also compare the investment levels of the sophisticated and naive type of agents: If the game ends in the first period, then both sophisticated and naive types attain the same investment level. If the game ends at a later period, $t = 2$ or $t = 3$, naive type always attains a higher overall investment level than the sophisticated type. Finally, we find that for the principal, it is

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[4]$\beta\delta$-preferences are first developed by Phelps and Pollak (1968) and later used by Laibson (1997), O'Donoghue and Rabin (1999a,1999b,2001) among others.

[5]We use Matlab to get these results, mainly because the model does not provide us with closed form solutions. The codes are available upon request.

better that the agent is naive rather than sophisticated for all δ and β values.

This paper is related to a number of other papers in the literature. Phelps and Pollak (1968) first developed the $\beta\delta$-preferences which were later used by Laibson (1997), O'Donoghue and Rabin (1999a, 1999b, 2001) among many others.[6] For instance, O'Donoghue and Rabin (1999a) examine self-control problem by using $\beta\delta$-preferences. Dellavigna and Malmendier (2004) investigate rational profit maximizing firms and consumers that have time-inconsistent preferences and they characterize the optimal contract that the firm offers for different types of time-consistent and time-inconsistent agents. Chade, Prokopovych and Smith (2008) study infinitely repeated games where players have $\beta\delta$-preferences. O'Donoghue and Rabin (1999b) consider principal-agent relationship with time-inconsistent agents. Eliaz and Spiegler (2006) characterize the optimal menu of contracts when a monopoly is contracting with time-inconsistent agents and show that it includes exploitative contracts for naive agents. In terms of investment decisions, especially regarding investment in education, one of the most influential paper is Spence (1973) which considers job market signaling where there is an asymmetric information between the employer and the employees. Employer cannot observe the productivity of the worker. However, by getting education, the employee signals his talent and this will mitigate the inefficiency resulting from the asymmetric information.

The organization of paper is as follows. In section 2, we briefly describe the model and introduce the types of agent. In section 3, we solve the model for time-consistent, time-inconsistent (sophisticated and naive) agent where we find the optimal streams for each type of agent for each δ, β combination. These results are depicted in the figures and we provide interpretations for them. In the same section, we also compare the overall investment levels of naive and sophisticated agent and answer the question with whom the principal is better off. In section 4, we briefly discuss the incomplete information case, where the principal does not have the information on the type of the agent. Section 5 concludes and provides further discussion. The technical details are given in the Appendix.

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[6]See also Gilpatric (2008), Akın (2009, 2012), Sarafidis (2006), Strotz (1956), Yan (2009a, 2009b) and Yılmaz (2013, 2015).

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2 The Model

There are two players, Agent and Principal, who interact over 3 periods.[7] The agent decides how much to invest and the principal observes this investment level and decides whether to offer a partnership or not. The agent is time-inconsistent captured through $\beta\delta$-preferences which are explained below, whereas the principal is time-consistent with the usual exponential discounting. Now, we give the details of the model.

2.1 Timing of the game

At time $t = 1$, the agent picks an investment level $\theta_1 \in [0; \infty)$ which has a cost $c(\theta_1)$. Then, the principal observes this investment level and decides whether to offer a partnership or not. Then, the agent decides whether to accept the offer or reject it. If the agent decides to accept, the production takes place and they get a total revenue of $R(\theta_1)$. The agent gets $sR(\theta_1)$ and the principal gets the remaining $(1-s)R(\theta_1)$, and the game ends. Otherwise, they move on to the next period.

At any other period $t > 1$, where there is no accepted offer in the previous period, the agent chooses an investment level $\theta_t \in [\theta_{t-1}; \infty)$ at a cost $c(\theta_t - \theta_{t-1})$. The principal observes this investment level and decides whether to offer a partnership or not. Then, the agent decides whether to accept the offer or reject it. If the agent decides to accept, the production takes place and they get a total revenue of $R(\theta_t)$. The agent gets $sR(\theta_t)$ and the principal gets the remaining $(1-s)R(\theta_t)$, and the game ends. Otherwise, they move on to the next period, unless $t = 3$. If, at the end of period $t = 3$, there is no partnership realized, no production takes place and no revenue generated.

2.2 Agent

The agent is time-inconsistent. A time-inconsistent agent discounts tomorrow to today by $\delta\beta$, but he discounts any future period to the previous period by only δ, from the viewpoint of today. Thus, his discount factor between two adjacent periods changes over time when he moves to the future periods. This implies that the agent's preferences are changing over time captured by

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[7]For the time-inconsistency to be effective, we need at least three periods.

quasi-hyperbolic discounting factors reflecting present-biased preferences.

A time-inconsistent agent can be either sophisticated (SO) or naive (NA). A sophisticated time-inconsistent agent uses quasi-hyperbolic discounting factors $(1, \delta\beta, \delta^2\beta)$ and knows (correctly) that at any future period he will be using this discounting scheme. Note that, the discount factor between today and tomorrow is $\delta\beta$, however the discount factor between tomorrow and the day after tomorrow is just δ, from today's point of view. A naive time-inconsistent agent, with discounting $(1, \delta\beta, \delta^2\beta)$, thinks (erroneously) that he will use $(1, \delta, \delta^2)$ at any future period.[8] We will analyze both types of agent separately.

As a benchmark we will consider a time-consistent agents who has the exponential discount factors $(1, \delta, \delta^2)$. Alternatively one can consider a time-consistent agent who has the exponential discount factors $(1, \delta\beta, (\delta\beta)^2)$. Note that for a time-consistent agent the discount factor between any period $t-1$ and t is constant. Hence the preferences do not change over time.

If the game ends at period T with an offer accepted, then the agent receives $sR(\theta_T)$ at the end of the game at period T, and incurs $c(\theta_t - \theta_{t-1})$ at each period $t \leq T$, where $1 \leq T \leq 3$ and $\theta_0 = 0$. If the game ends at $t = 3$ with no offer or with an offer rejected, then no revenue is realized, and the agent only incurs the cost of investment stream. The strategy set of the agent is given by

$$S_A = \{s_A^1 : [0, \infty) \times \{A, R\}\} \times \{s_A^2 : [0, \infty) \to [0, \infty) \times \{A, R\}\} \times \{s_A^3 : [0, \infty)^2 \to [0, \infty) \times \{A, R\}\}$$

where A stands for Accept and R stands for Reject.

2.3 Principal

The principal is time-consistent and has the usual exponential discount factors $(1, \delta, \delta^2)$. If the game ends at period T with an offer accepted, then the principal receives $(1 - s)R(\theta_T)$ at the end of the game at period T, where $1 \leq T \leq 3$, and incurs no cost. If the game ends at $t = 3$ with no offer or with an offer rejected, then no revenue is realized and the principal receives

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[8]A naive time-inconsistent agent might be either fully naive or partially naive. The former case is when the agent thinks his β is equal to 1, and the latter case is when he thinks his β is not equal to 1 but still larger than the true value. We consider the former case in this paper.

zero, his reservation payoff. The strategy set of the principal is given by

$$S_P = \{s_P^1 : [0, \infty) \to \{P, N\}\} \times \{s_P^2 : [0, \infty)^2 \to \{P, N\}\} \times \{s_P^3 : [0, \infty)^3 \to \{P, N\}\}$$

where P stands for Partnership, and N stands for No Partnership.

3 Complete Information

In this section, we assume that all of the above is common knowledge and that the principal knows the agent's cost function and whether he is time-consistent or sophisticated time-inconsistent or naive time-inconsistent. We also assume specific revenue and cost functions: $R(\theta) = \theta$ and $c(\theta) = \theta^2$.

3.1 Benchmark: Time-consistent Agent

We assume that the agent is time-consistent with the discounting factor δ. We find the optimal investment streams that would end the game in periods $t = 1, 2, 3$, respectively. First, we start with the optimal payoff stream that would end the game at $t = 3$. Given θ_2, in the third period the principal will make an offer for any $\theta_3 \geq 0$ and the agent maximizes $s\theta_3 - (\theta_3 - \theta_2)^2$ with respect to θ_3. The agent's optimal investment level is $\theta_3^{***} = \theta_2^{***} + s/2$. The optimal level of investment at the second period (in order to end the game at the third period) is equal to $\theta_2^{***} = \theta_1^{***} + s\delta/2$.[9] Then, the optimal θ_1^{***} can be find by maximizing the overall payoff stream at period 1 respect to θ_1. So, the maximization problem is given by

$max_{\theta_1} \; \delta^2[s\theta_3 - (\theta_3 - \theta_2)^2] - \delta[\theta_2 - \theta_1]^2 - \theta_1^2$ where $\theta_3 = \theta_2 + s/2$ and $\theta_2 = \theta_1 + s\delta/2$,

which gives $\theta_1^{***} = \delta^2 s/2$. As a result the optimal stream that will take the agent to the 3rd period is

$$\theta_1^{***} = \frac{\delta^2 s}{2} \quad \theta_2^{***} = \frac{\delta s}{2}(1 + \delta) \quad \theta_3^{***} = \frac{s}{2}(\delta^2 + \delta + 1)$$

Now, we find the optimal stream of investments that ends the game at the second period. At $t = 2$, the agent maximizes $s\theta_2 - (\theta_2 - \theta_1)^2 - \theta_1$ with respect to θ_2 and this maximization gives $\theta_2^{**} = \theta_1 + s/2$. The optimal investment level at $t = 2$ to end the game at $t = 2$ is $s/2$, but

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[9]See the appendix

this investment level will not be enough to get an offer. To get an offer it must be $\theta_2 \geq \frac{s\delta}{2(1-\delta)}$. Hence, it must be $\theta_2^{**} = \theta_1 + s/2 \geq \frac{s\delta}{2(1-\delta)}$, otherwise θ_2^{**} will not be enough to get offer. This situation will occur when θ_1 is very small and hence the additional investment is not enough to get an offer. In these cases, the agent will choose the (threshold) investment level that receives an offer, which is $\frac{s\delta}{2(1-\delta)}$. So,

$$\theta_2^{**} = \begin{cases} \frac{s\delta}{2(1-\delta)} & \text{if } \theta_1 < \frac{(2\delta-1)s}{2(1-\delta)} \\ \theta_1^{**} + s/2 & \text{if } \theta_1 \geq \frac{(2\delta-1)s}{2(1-\delta)} \end{cases}$$

Given θ_2^{**}, the optimal level of θ_1^{**} can be find by maximizing

$$\Pi_A^2 = \delta[s\theta_2 - (\theta_2 - \theta_1)^2] - \theta_1^2$$

which gives the following optimal investment level [10]

$$\theta_1^{**} = \begin{cases} \frac{s\delta^2}{2(1-\delta)(1+\delta)} & 1 < \delta^2 + \delta \\ \frac{s\delta}{2} & 1 \geq \delta^2 + \delta \end{cases}$$

Hence the optimal stream that ends the game in the second period is

$$(\theta_1^{**}, \theta_2^{**}) = \begin{cases} \{\frac{s\delta^2}{2(1-\delta)(1+\delta)}, \frac{\delta s}{2(1-\delta)}\} & 1 < \delta^2 + \delta \\ \{\frac{s\delta}{2}, \frac{s(1+\delta)}{2}\} & 1 \geq \delta^2 + \delta \end{cases}$$

Finally, the optimal investment level that ends the game in the first period is [11]

$$\theta_1^* = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > \delta^2 + \delta \\ \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}$$

Now, for any given δ, we find which period the game ends, thus the optimal investment levels.

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[10]See the Appendix.
[11]See the Appendix.

Proposition 1 *Suppose the agent is time-consistent with a discount factor δ. Then, for any given δ, the equilibrium investment levels are given by*

$$\theta^* = \theta_1^* \qquad\qquad\qquad\qquad if \ \delta \le \underline{\delta}$$

$$\theta^{**} = (\theta_1^{**}, \theta_2^{**}) \qquad\qquad\qquad if \ \underline{\delta} < \delta \le \overline{\delta}$$

$$\theta^{***} = (\theta_1^{***}, \theta_2^{***}, \theta_3^{***}) \qquad\qquad if \ \overline{\delta} < \delta$$

where $\theta^*, \theta^{**}, \theta^{***}$ are the optimal investment streams that end the game in periods 1,2,3 respectively.

Proof. See the Appendix. ∎

For low levels of δ, it is optimal to pick an investment level that receives an offer in the very first period and the agent accepts it. For intermediate levels of δ, it is optimal to pick a stream of investment levels that ends the game in the second period. Note that, for this investment stream, if the investment level in the first period receives an offer, it is optimal for the agent to reject it. Finally for high levels of δ, it is optimal to pick an investment stream in order to end the game in the last period. The intuition for this result is straightforward: As δ increases, the future periods are getting more valuable, and hence it is optimal to spread the overall investment over two or more periods which is less costly, particularly because the cost function is quadratic.

3.2 Sophisticated Agent

We assume that the agent is time-inconsistent with the discounting scheme $(1, \delta\beta, \delta^2\beta)$. We also assume that the agent is aware of his discounting, that is, he is sophisticated. We find the optimal investment streams that would end the game in periods $t = 1, 2, 3$, respectively. Then for each (δ, β) pair, we will find which stream of investments is actually optimal.

First, we find the optimal decisions at $t = 3$. In this last period, the principal will make an offer for any investment level that is bigger than zero, his reservation utility. Given θ_2, the agent maximizes $s\theta_3 - (\theta_3 - \theta_2)^2$ with respect to θ_3, and the optimal level is $\theta_3^{***} = \theta_2^{***} + s/2$. Given this optimal investment level in the last period, the optimal level of investment in the second period in order to end the game in the third period is equal to $\theta_2^{***} = \theta_1^{***} + s\delta\beta/2$.[12] Now, given

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[12]See the Appendix.

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these optimal levels in the second and the third periods, we solve for optimal strategy in the first period (aiming to take the game to the last period). This optimal level, θ_1^{***}, can be find by solving the following maximization problem, $max_{\theta_1} \ \delta^2\beta[s\theta_3 - (\theta_3 - \theta_2)^2] - \delta\beta[\theta_2 - \theta_1]^2 - \theta_1^2$ where $\theta_3 = \theta_2 + s/2$ and $\theta_2 = \theta_1 + s\delta\beta/2$. The optimal level is then $\theta_1^{***} = s\delta^2\beta/2$. Now plugging this θ_1^{***}, we will get θ_2^{***} and θ_3^{***}. As a result the optimal stream, conditional on the game ending in the third period, is

$$\theta_1^{***} = \frac{s\delta^2\beta}{2}, \quad \theta_2^{***} = \frac{s\delta\beta}{2}(1 + \delta), \quad \theta_3^{***} = \frac{s}{2}(\delta^2\beta + \delta\beta + 1)$$

Now we find the optimal stream conditional on the game ends in the second period. In the second period, the agent maximizes $s\theta_2 - (\theta_2 - \theta_1)^2 - \theta_1$ with respect to θ_2 and this maximization gives $\theta_2^{**} = \theta_1 + s/2$. The optimal investment level at $t = 2$ to make the game end at $t = 2$ is simply $s/2$, but this additional investment will not be enough to get an offer. To get an offer, it must be $\theta_2 \geq \frac{s\delta}{2(1-\delta)}$. Hence, θ_2 must satisfy the condition $\theta_2^{**} = \theta_1 + s/2 \geq \frac{s\delta}{2(1-\delta)}$, otherwise with θ_2^{**}, the agent will not get an offer, which is the case when θ_1 is very small. In these cases, the agent will choose the (threshold) investment level that receives an offer, which is $\frac{s\delta}{2(1-\delta)}$. Thus, we get

$$\theta_2^{**} = \begin{cases} \frac{s\delta}{2(1-\delta)} & \text{if } \theta_1 < \frac{(2\delta-1)s}{2(1-\delta)} \\ \theta_1^{**} + s/2 & \text{if } \theta_1 \geq \frac{(2\delta-1)s}{2(1-\delta)} \end{cases}$$

Given θ_2^{**}, the optimal level of level θ_1^{**} can be find by maximizing the following.

$$\Pi_A^2 = \delta\beta[s\theta_2 - (\theta_2 - \theta_1)^2] - \theta_1^2$$

which gives the following optimal investment levels [13]

$$\theta_1^{**} = \begin{cases} \frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)} & 1 < 2\delta + \delta^2\beta - \delta\beta \\ \frac{s\delta\beta}{2} & 1 \geq 2\delta + \delta^2\beta - \delta\beta \end{cases}$$

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[13]See the Appendix.

Hence, the optimal stream conditional on the game ending in the second period is

$$(\theta_1^{**}, \theta_2^{**}) = \begin{cases} \{\frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)}, \frac{\delta s}{2(1-\delta)}\} & 1 < 2\delta + \delta^2\beta - \delta\beta \\ \{\frac{s\delta\beta}{2}, \frac{s(1+\delta\beta)}{2}\} & 1 \geq 2\delta + \delta^2\beta - \delta\beta \end{cases}$$

Lastly, the optimal investment level conditional on the game ends in the first period is given by [14]

$$\theta_1^* = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > 2\delta^2 + \delta^3\beta \\ \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}$$

Given the optimal investment streams conditional on ending the game at periods $t = 1, 2, 3$, we will now determine which of the three streams is actually optimal, for any given (δ, β) pair. However, unlike the case where the agent is time-consistent, here we cannot just compare the three overall payoff levels of these three options simply because the discounting is changing over time. Thus, we will first determine, for any (δ, β) pair, the optimal investment levels given the game has reached $t = 2$, and thus determine which period to end the game. When doing this, we use $\delta\beta$ to discount period 3 to period 2. This is because, sophisticated agent knows, at period 1, that, at period 2 he will discount period 3 to period 2 by $\delta\beta$. Then, having determined what is optimal in the second period, we compare the overall payoff levels that arise when the game ends at $t = 1$ with the maximized continuation payoff. When comparing these two, we use the discounting $(1, \delta\beta, \delta^2\beta)$ because now everything is from the viewpoint of the first period.

In the Figure 1, we provide the analysis for $s = 0.7$. Although the investment levels and overall profit levels depend on s, the profit ranking of ending the game in different periods does not depend on s. For any given (δ, β) pair, the figure depicts the period at which the game optimally ends.
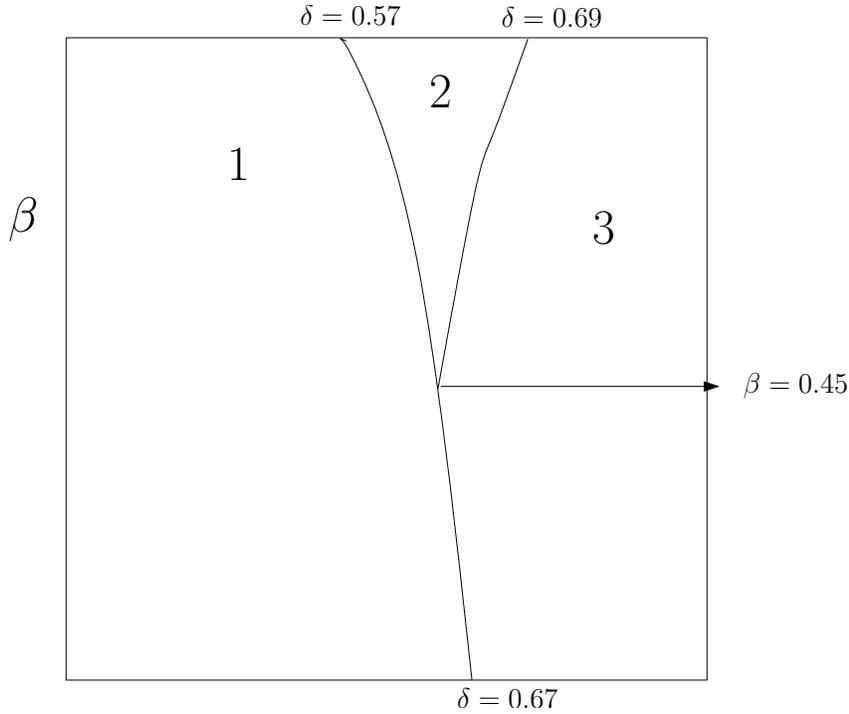
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[14]See the Appendix.

Figure 1 : The periods at which the game ends for different δ-β pairs for sophisticated agent, for $s = 0.7$.

Interpretation of Figure 1:

We have four observations in Figure 1 regarding both δ and β which reflect the degree of time-inconsistency. (1) For any given β which is bigger than 0.45, as δ increases, the optimal period to end the game increases. (2) For $0.57 < \delta < 0.65$, as β increases, the agent aims to end the game in the second period rather than the first period. (3) For $0.69 > \delta \geq 0.65$, as β increases, the agent aims to end the game in the second period rather than the third period. (4) For low values of β the game never finalizes in the second period. The intuition behind these observations are as follows.

(1) Fixing β where $\beta > 0.45$, an increase in δ means that the discounting increases hence, the latter periods are relatively more valuable. And, the agent is more willing to spread out total investment over t=1,2,3. Because cost function is quadratic, making investment all at once is more costly than spreading over time. Moreover, for higher δ, the revenue attained at a future period is also more valuable from the viewpoint of today. Hence by the impact of these two effects, due to an increase in δ, it becomes optimal to spread the overall investment over three periods. Also note that for $\beta = 1$, this figure gives us the picture for the time-consistent agent.

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(2) For $0.57 < \delta < 0.65$, as β increases, it is optimal to choose an investment stream that ends the game in the second period instead of ending the game in the first period. For example, for $\delta = 0.6$, the agent will attain the same investment level for terminating the game at t=1 and t=2, that is, $\theta_1 = \theta_2$. The intuition is that although the investment levels are the same, ending game at t=2 is less costly because spreading investment over more periods decreases cost due to the quadratic cost function. Although the revenue is bigger when terminating the game at t=1 ($s^2\theta_1 > \delta\beta(s^2\theta_2)$), the difference is getting smaller as β increases, and when $\beta = 1$ the difference vanishes. However, the cost is smaller for terminating the game at t=2, because, the agent divides the same investment level into two parts and spreads it over two periods.[15]

(3) For $0.69 > \delta \geq 0.65$, when β is high, the game ends at t=2, when β has a medium value the game finalizes at t=3 and for very small values, the game ends at t=1. The figures 2 and 3 are drawn for $\delta = 0.66$. As can be seen, for $\beta < 0.6$, the game ends at t=3, and for $\beta > 0.6$ the game ends at t=2, and for $\beta < 0.2$ the game ends at first period. The reason why the terminal period moves from the third period to the second period is related to the cost structure in the second period. The cost increases with a decreasing β for $\beta > 0.6$, and the revenue is linear for this region because the investment level is equal to $\frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)}$ and hence revenue of ending the game in the second period dominates the revenue of ending the game in the third period. Moreover, The agent's profit in the third period, by the viewpoint of the second period is

$$\delta\beta[s\theta_3 - (\theta_3 - \theta_2)^2] - [\theta_2 - \theta_1]^2$$

As β increases, the revenue ($\delta\beta s\theta_3$) increases quadratically (through θ_3). However, the cost ($\delta\beta[(\theta_3-\theta_2)^2]+[\theta_2-\theta_1]^2$) also increases quadratically. Hence, the cost and the revenue functions work in opposite ways. As β increases, the agent will become more eager to finish the game in the third period, since the revenue increases, but the cost also increases and the agent finds it optimal to finish the game in the second period. Note that, in the observation (2) above, the two effects work in the same direction. Hence, as β increases, the agent becomes more eager to finish game in the second period rather than in the first period.

Second period's profit is also bigger than the first period's profit for this region. For instance,

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[15]See the Appendix.

for $\delta = 0.66$, the investment levels are $\frac{\delta s}{2(1-\delta)} = \theta_1^* = \theta_2^{**} > \theta_3^{***}$. The overall investment levels for the first period and for the second period are the same, but the cost of ending the game in the second period is well below that of the first period. This is due to the fact that spreading the same investment level over two periods rather than one period is less costly. Revenue is also smaller in the second period than that of the first period due to the fact that the agent discounts the same revenue level to the previous period by $\beta\delta$. Hence for low values of β, it is optimal to finalize the game in the first period, but for high values of β finalizing the game in the second period is optimal.

(4) For low values of β, when δ is high, the game ends at t=3, and when δ is low the game finalizes at t=1. And the game never finalizes in the second period. This is because for low values of δ and β, the agent discounts the future too much and it is optimal to finalize the game in the first period. For $\delta < 0.5$, he chooses $\theta_1^* = \frac{s}{2}$, that is why the profit function is linear in this region. However, for $\delta > 0.5$, he chooses $\theta_1^* = \frac{s\delta}{2(1-\delta)}$ and the profit function strictly decreases in this region. Hence, after some δ, the profit levels of ending the game in the second and third period dominate that of ending the game in the first period, because as δ increases future becomes more valuable. However, second period profit is dominated by the first period profit in region 1 and dominated by the third period profit in region 3, and this is illustrated in Figure 4 and 5.

The following result is directly observed in Figure 1.

Proposition 2 *Suppose the agent is time-inconsistent and sophisticated with a discount scheme* $(1, \delta\beta, \delta^2\beta)$. *Then, there are threshold values,* $\delta_1, \delta_2, \delta_3, \delta_4$ *such that, in the equilibrium, for any* (δ, β) *with*

(i) $\delta < \delta_1$, the game ends in the first period,

(ii) $\delta_1 < \delta < \delta_2$, for low β values the game ends in the first period, and for high β values the game ends in the second period,

(iii) $\delta_2 < \delta < \delta_3$, for low β values the game ends in the first period, and for medium β values the game ends in the third period, and for high β values the game ends in the second period,

(iv) $\delta_3 < \delta < \delta_4$, for low β values the game ends in the third period, and for high β values the game ends in the second period,

(v) $\delta_4 < \delta$, *the game ends in the third period.*

3.3   Naive Agent

In this section we assume that the agent is time-inconsistent with a discounting scheme $(1, \delta\beta, \delta^2\beta)$ and moreover, he is not aware of this discounting scheme. More specifically, in the first period naive agent thinks that he is time-consistent and thus uses $(1, \delta, \delta^2)$, and thinks that he will be using this scheme in the future. So, in the first period, he thinks he will discount period 3 to period 2 by δ, however, when he arrives at period 2, he learns he is time-inconsistent and his discounting scheme is actually $(1, \delta\beta, \delta^2\beta)$, thus discounts period 3 to period 1 by $\delta\beta$. Hence, from the point view of the first period, he solves the problem as a time-consistent agent with a discount factor, δ. As a result, the solution is the same as the time-consistent agent's optimal streams

$$\theta_1^{***} = \frac{\delta^2 s}{2} \quad \theta_2^{***} = \frac{\delta s}{2}(1 + \delta) \quad \theta_3^{***} = \frac{s}{2}(\delta^2 + \delta + 1)$$

$$(\theta_1^{**}, \theta_2^{**}) = \begin{cases} \{\frac{s\delta^2}{2(1-\delta)(1+\delta)}, \frac{\delta s}{2(1-\delta)}\} & 1 < \delta^2 + \delta \\ \{\frac{s\delta}{2}, \frac{s(1+\delta)}{2}\} & 1 \geq \delta^2 + \delta \end{cases}$$

$$\theta_1^{*} = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > \delta^2 + \delta \\ \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}$$

which end the game in the third, the second and the first periods, respectively.

Thus, depending on his δ, he makes one of the initial investments levels, θ_1^{*} or θ_1^{**} or θ_1^{***}. However, in the case where he chooses one of the θ_1^{**} or θ_1^{***}, the game moves on to the second period and he learns his type. At that moment, he might realize that the investment levels he was planning to choose may not be optimal anymore. So, he recalculates his optimal investment level(s) from that point on. The agent, for a given δ (that made him pick an investment level aiming to end the game in the second or third period), has two options in the second period: either aim to end the game in the period he initially intended or change his mind and end the game in the other period (in the second period if he intended for the third period, or the third period if he intended to end it in the second period). We look at these two possible cases

separately: (1) Suppose he invests θ_1^{**} and plan to end the game in the second period. Now, he calculates the optimal stream that will take him to the third period.[16] Here, we need to consider two different regions. One region is where $1 < \delta^2 + \delta$. For this region, we have

$$\theta_1^{**} = \frac{s\delta^2}{2(1-\delta)(1+\delta)}, \quad \theta_2^{***} = \frac{s\delta^2}{2(1-\delta)(1+\delta)} + \frac{\delta\beta s}{2}, \quad \theta_3^{***} = \frac{s\delta^2}{2(1-\delta)(1+\delta)} + \frac{\delta\beta s}{2} + \frac{s}{2}$$

The other region is where $1 \geq \delta^2 + \delta$. For this region, we have

$$\theta_1^{**} = \frac{s\delta}{2}, \quad \theta_2^{***} = \frac{\delta\beta}{2} + \frac{s\delta}{2}, \quad \theta_3^{***} = \frac{\delta\beta s}{2} + \frac{s\delta}{2} + \frac{s}{2}$$

(2) Now suppose he invests θ_1^{***} and plan to end the game in the third period. Now, he calculates the optimal stream that will terminate the game in the second period.[17]

$$\theta_1^{***} = \frac{\delta^2 s}{2} \quad \theta_2^{**} = \begin{cases} \frac{s(1+\delta^2)}{2} & 1 \geq \delta^3 + 2\delta - \delta^2 \\[2mm] \frac{s\delta}{2(1-\delta)} & 1 < \delta^3 + 2\delta - \delta^2 \end{cases}$$

In the light of these investment levels, Figure 2 below depicts the periods that the game will terminate for any pair of δ, β. There is a region where he changes his mind and finishes the game in the second period instead of the third period. Moreover, at that particular region he realizes that he had made a mistake in his investment decision in the first period, that he chose θ_1^{***} erroneously instead of θ_1^{**} which he must have chosen if he were aware of his time-inconsistency.

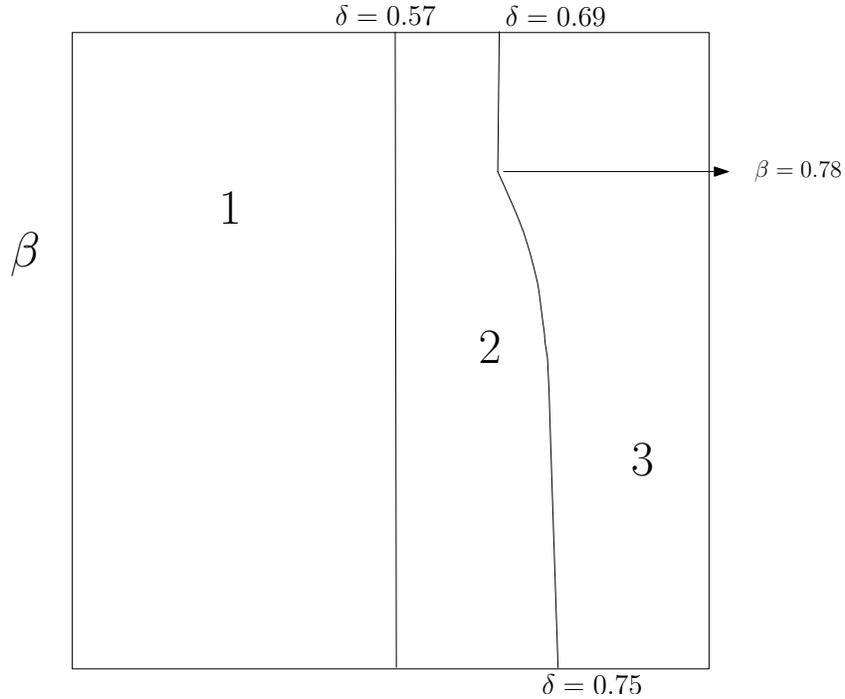[16]See the Appendix.
[17]See the Appendix.

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Figure 2: The periods at which the game ends for different δ-β pairs for naive agent for $s = 0.7$.

Interpretation of Figure 2:

As can be seen in Figure 2, for some δ and β combinations, the agent changes his mind after learning about his inconsistency and decides to end the game in the second period rather than the third period. For other δ and β combinations, however, there is no change. For example, for $\beta \geq 0.78$, there is no change in the agent's decision (about which period to aim to end the game) even after he learns about his time-inconsistency. For these high values of β, his time-inconsistency is small, and he had chosen an investment level that is very close to the investment level he picks after he learns his true type. Moreover for $\beta = 1$, he makes the same investment as the time-consistent agent. As shown in Figure 2, for $\delta > 0.75$, the profit in the third period is larger than the one in the second period.

There is a region where the agent, after learning his true β, changes his mind about his investment decision. For example, for $\delta = 0.73$, for low β values, he decides to end the game in the second period instead of aiming to end the game in the third period. For such (δ, β) pairs, his intended period to finalize the game, namely the third period, is not optimal anymore once he observes his true β. He will realize that he is discounting future more than he thinks. Hence, the future periods aren't favorable anymore, then he ends the game in the second period.

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Moreover, as β decreases, the range of δ values expands where he changes his mind. This is simply because for such βs, he is discounting future too much.

The following result is directly observed in Figure 1.

Proposition 3 *Suppose the agent is time-inconsistent and naive with a discount scheme $(1, \delta\beta, \delta^2\beta)$. Then, there are threshold values, $\delta_1, \delta_2, \delta_3$ such that, in the equilibrium, for any (δ, β) with*

(i) $\delta < \delta_1$, the game ends in the first period,

(ii) $\delta_1 < \delta < \delta_2$, the game ends in the second period,

(iii) $\delta_2 < \delta < \delta_3$, for high β values, the game ends in the third period, and for low β values, the game ends in the second period, where he initially intended to end the game in the third period,

(iv) $\delta_3 < \delta$, the game ends in the third period.

3.4   Comparison: Overall Investment Levels

In this section, we look at all optimal investment streams that end the game in periods t=1,2,3 for both naive and sophisticated agents, and then we compare them. We have the following observations directly from the figure below.
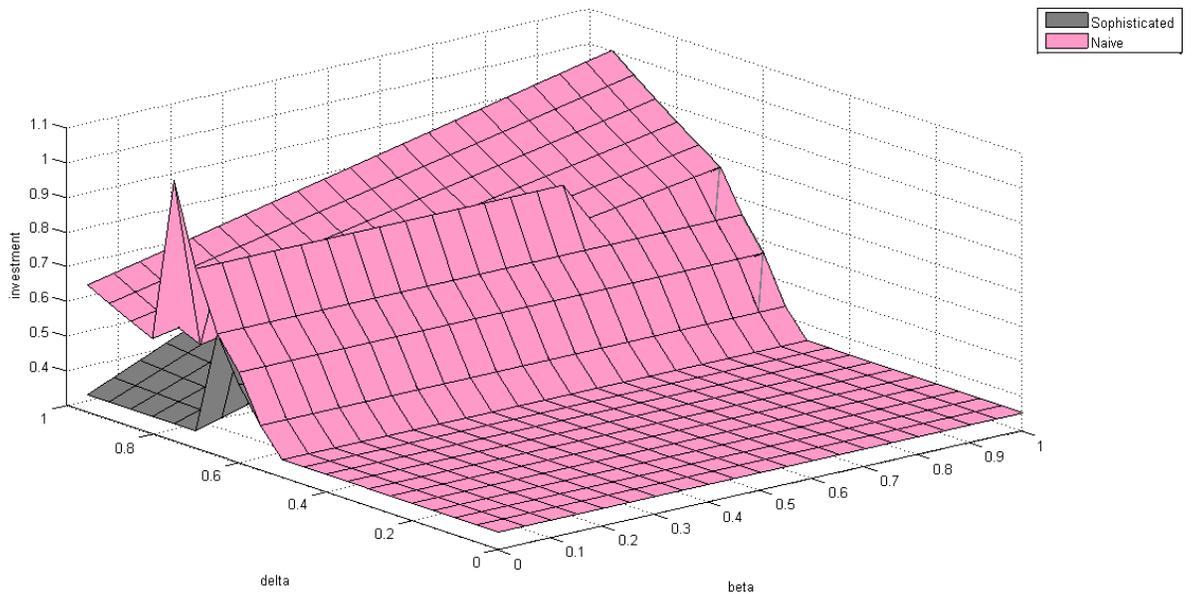

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Figure 3: Investment levels for both sophisticated and naive agents for $s = 0.7$.

(1) For those (δ, β) combinations where they end the game in t=1, both naive and sophisticated agents pick the same investment level, because in the first period they do not use any discount factor. Hence optimization gives the same investment level which is equal to $\frac{s}{2}$, and this investment level gets an offer for $\delta < 0.5$. As a result, for this region investment level is linear. However, $\frac{s}{2}$ is not an enough to get an offer for $\delta > 0.5$. Hence, for such (δ, β) combinations, the agent chooses an investment level that is the minimum amount to get an offer, which is $\frac{s\delta}{2(1-\delta)}$.

(2) For all (δ, β) combinations, the naive agent attains an investment level that is at least as big as the investment level of the sophisticated agent.[18] Intuitively, the reason that the naive agent picks a higher investment level is that he is not aware his time-inconsistency and he thinks he is time-consistent, and he does not know he is going to discount future more than he thinks. And hence he is overconfident about future and he picks a higher investment today. However, the sophisticated agent is totally aware of his time-inconsistency problem, and does not suffer from over-investment problem in the initial period.

3.5   With whom the Principal is better off?

Principal's profit is equal $\Pi_P = (1 - s)\theta$, and it increases in the investment level agent attains. Hence, the principal is better off with the agent who attains a higher overall investment level. The principal will weakly prefer naive agent over sophisticated agent for all (δ, β). If the agent is naive, the firm can exploit consumer's naivete and can get more out of him because naive agent is not aware of his time-inconsistency, namely his true β. Hence, he erroneously makes a high initial investment, $(\theta_1^{**}, \theta_1^{***})$ as compared to the sophisticated agent's initial investment level. And the difference in investment levels increases as $(1 - \beta)$ increases, that is, as his naivete increases.[19] This is summed up in the following proposition.

Proposition 4 *For all (δ, β) combinations, the naive agent attains an overall investment level at least as big as that of the sophisticated agent, and the principal is weakly better off with a naive agent. For a range of (δ, β) combinations, the principal is strictly better off with the naive agent.*

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[18]See the Appendix.

[19]See the Appendix.

4 Discussion

Incomplete Information:

There are several ways to extend our model to the incomplete information environment. First, we can assume that the agent has more than one possible cost function and the principal does not know which cost structure the agent will actually use. For instance, we can assume the cost is either $c(\theta) = (\frac{\theta}{2})^2$ or $c(\theta) = \theta^2$. In this case the first period investment levels are going to serve as signals and the principal will update his beliefs about the agent's type (high cost or low cost). We need to use Perfect Bayesian Nash equilibrium as the solution concept and find all pooling and separating equilibria. However, for a pooling equilibrium not only both types will need to pick the same investment level in the first period, they should also pick the same in the second and the third periods. In a separating equilibrium, it must be the case that the two types will pick different levels of investments and their types will be revealed, hence from then on the principal will know who he is facing. This analysis turned out to be highly complicated and we leave it for further research.

An alternative way of incorporating incomplete information is to assume that the principal knows that the agent is time-inconsistent but he does not know whether the agent is sophisticated or naive. However, nature telling the agent that he is with some positive probability naive will involve some information content and hence will contradict the definition of naivete. By providing the agent with this information, namely his type space, we change his information content, hence we change his type space.

Even if we can find a way to expose incomplete information into our setting, it does not help principal to learn the agent's type (that is, whether it is naive or sophisticated) since the principal's offer thresholds are the same for both naive and sophisticated agent. Moreover, if the agent wants to finish in the first period, principal will fail to get new information because both naive and sophisticated will definitely choose the same investment level, and the principal will make an offer, and then, the game ends. Furthermore, there will be a problem if the agent aims to finalize the game in the third period. Revealing his type gives no advantage to the principal, even if the principal makes an offer, agent will reject the offer and is going to end up in the third period.

Endogenous sharing rule:

We can think of a sharing rule that depends on the investment level, $s(\theta)$. Plausible restrictions on such a sharing rule are the following: $s(0) = 0$, $s(\infty) = 1$, $s' > 0$, $s'' < 0$. For instance,

$$s_1(\theta) = \frac{\theta}{\theta + 1}$$

$$s_2(\theta) = \begin{cases} \frac{\theta}{2} & \theta \in [0, 1) \\[2mm] 1 - \frac{1}{2\theta} & \theta \in [1, \infty) \end{cases}$$

are two possible sharing rules that satisfy those conditions we just mentioned above. We tried the first one, however, the solution turns out to be a lot more complicated and hence it is not easy to get the optimal investment streams.

More general functional forms:

Although the revenue and cost functions we used are specific, they are plausible ones and we believe that using more general functions will not considerably affect our results.

5 Conclusion

In this paper, we looked at a three period dynamic investment model where there is a time-inconsistent agent and a time-consistent principal. The time-inconsistency is captured by $\beta\delta$ preferences. We considered both cases where the agent is aware of his time-inconsistency and where the agent is not aware of his inconsistency but learns his true β in the beginning of the second period. However, sophisticated agent knows his true β in the beginning of the first period. This difference in their information set is causing them to finalize the game at different periods or to attain different level of investments. We found that for all β, δ combinations, the naive agent ends up investing a higher (or equal) overall amount than the sophisticated agent, and thus the principal is (weakly) better off with a naive agent.

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6 Appendix

6.1 Time-consistent Agent

(1) To get the optimal level of investment in the second period in order to end the game in the third period, he maximizes $\delta[s\theta_3 - (\theta_3 - \theta_2)^2] - (\theta_2 - \theta_1)^2$ where $\theta_3 = \theta_2 + s/2$. Then, $\max_{\theta_2} \delta[s\theta_2 - s^2/2 - s^2/4] - (\theta_2 - \theta_1)^2$ gives optimal θ_2^{***} which is optimal investment level that ends the game in the third period: $\theta_2^{***} = \theta_1^{***} + s\delta/2$.

(2) To get the optimal stream of investment that will end game in the second period, define $\frac{(2\delta-1)s}{2(1-\delta)} = K$.

Then, if $\theta_1 < \frac{(2\delta-1)s}{2(1-\delta)} = K$, we have $\theta_2 = \frac{s\delta}{2(1-\delta)}$. Then, solving

$$max_{\theta_1} \delta[s\theta_2 - (\theta_2 - \theta_1)^2] - \theta_1^2$$

we get $\delta(\theta_2 - \theta_1) = \theta_1$, which implies

$$\frac{\delta\theta_2^*}{1 + \delta} = \theta_1^*$$

Thus, we get

$$\theta_1^* = \frac{s\delta^2}{2(1 - \delta)(1 + \delta)}$$

We also need to check whether this θ_1^* is smaller than K or not, that is,

$$\frac{s\delta^2}{2(1 - \delta)(1 + \delta)} < \frac{(2\delta - 1)s}{2(1 - \delta)} \quad \text{which implies} \quad 1 < \delta^2 + \delta.$$

For the other case where $\theta_1 \geq K$ we have $\theta_2 = \theta_1 + s/2$. Then solving

$$max_{\theta_1} \delta[s(\theta_1 + \frac{s}{2}) - (\theta_2 - \theta_1)^2] - \theta_1^2$$

we get $\delta s = 2\theta_1$ which implies $\theta_1^* = \delta s/2$. Thus, the optimal stream of investment is

$$(\theta_1^{**}, \theta_2^{**}) = \begin{cases} \{\frac{s\delta^2}{2(1-\delta)(1+\delta)}, \frac{\delta s}{2(1-\delta)}\} & 1 < \delta^2 + \delta \\ \{\frac{s\delta}{2}, \frac{s(1+\delta)}{2}\} & 1 \geq \delta^2 + \delta \end{cases}$$

(3) Now we find the optimal investment level to end game in the first period.

First of all we must guarantee that the investment level θ_1 gets an offer in the first period, which means that it must give higher payoff to the principal in the first period than the payoff principal gets in the second and third period. Payoffs of principal from the first period point of view is are as follows. In the first period, she gets $(1-s)\theta_1$, in the second period she gets $\delta(1-s)\theta_2$, and in the third period she gets $\delta^2(1-s)\theta_3$.

Suppose $\theta_1 < K$. We will check the payoff for principal and see whether the first period is optimal or not to offer a partnership, that is, $\Pi_1^p > \Pi_2^p$, $\Pi_1^p > \Pi_3^p$ both hold. For the first inequality we need $\theta_1 > \frac{s\delta^2}{2(1-\delta)}$, which is obtained using $\theta_2^{**} = \frac{\delta s}{2(1-\delta)}$ and $(1-s)\theta_1 > \delta(1-s)\theta_2$. For the second inequality we need $\theta_1 < \frac{\delta^2 s(1+\delta)}{2(1-\delta^2)}$, which is obtained from $\theta_3^{***} = \theta_1 + \frac{s}{2} + \frac{s\delta}{2}$ and $\delta^2(1-s)(\theta_3) < (1-s)\theta_1$.

However the implication from the first condition contradicts with $\theta_1 < K$ since $\frac{(2\delta-1)s}{2(1-\delta)} > \theta_1 > \frac{s\delta^2}{2(1-\delta)}$ implies $0 > (\delta-1)^2$ which is a contradiction. Hence ending in the first period is not profitable for principal for the this first case.

Now suppose, $\theta_1 \geq K$. We will again check $\Pi_1^p > \Pi_2^p$ and $\Pi_1^p > \Pi_3^p$. The first condition implies $\theta_1 > \frac{\delta s}{2(1-\delta)}$ since $\theta_2^{**} = \theta_1 + s/2$ and $(1-s)\theta_1 > \delta(1-s)\theta_2$. The second condition implies $\theta_1 > \frac{\delta^2 s}{2(1-\delta)}$ since $\delta^2(1-s)(\theta_3) < (1-s)\theta_1$ and $\theta_3^{***} = \theta_1 + \frac{s}{2} + \frac{s\delta}{2}$.

However, we have $\frac{\delta s}{2(1-\delta)} > \frac{\delta^2 s}{2(1-\delta)}$, thus $\frac{\delta s}{2(1-\delta)}$ is the minimum amount that guarantees to get the partnership offer when we compare the second and third period payoffs with that of the first period for the principal. Hence, the optimal amount of investment to finish the game in first is as follows:

$$
\theta_1^{**} = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > \delta^2 + \delta \\[2mm] \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}
$$

Proof of Proposition 1:

Ending the game in periods $t = 1, 2, 3$ gives the following profit levels for the agent, respectively.

$$\Pi_A^1 = \begin{cases} \frac{s^2}{4} & \frac{1}{2} > \delta \text{ and } 1 > \delta^2 + \delta \\\\ \frac{s^2\delta}{2(1-\delta)} - (\frac{s^2\delta}{2(1-\delta)})^2 & \text{otherwise} \end{cases}$$

$$\Pi_A^2 = \begin{cases} \delta[\frac{\delta s^2}{2(1-\delta)} - (\frac{s\delta}{2(1-\delta)(1+\delta)})^2] - (\frac{s\delta^2}{2(1-\delta)(1+\delta)})^2 & 1 < \delta^2 + \delta \\\\ \delta[\frac{s^2(1+\delta)}{2} - \frac{s^2}{2}] - \frac{s^2\delta^2}{4} & 1 \geq \delta^2 + \delta \end{cases}$$

$$\Pi_A^3 = \delta^2[\frac{s^2}{2}(\delta^2 + \delta + 1)] - \frac{s^2}{4} - \delta[\frac{\delta s}{2}]^2 - \frac{\delta^4 s^2}{4}$$

We find these thresholds $(\underline{\delta}, \overline{\delta})$ by Matlab.[20] The algorithm works as follows. First it compares Π_A^3 and Π_A^2 from the point of view of the second period by treating the first periods investment level as a sunk cost. Hence we find two regions on δ, where the game ends in the second period and end in the third period. Then, we calculate Π_A^1 and first we compare it to the payoff if the game ends in the third period profit, from the point of view of the first period, and we get a third region. However, for this region optimal stream of investment levels that send the game to the third period gives higher profit, hence $\overline{\delta}$ is determined by comparing the payoffs of the second and the third period. Second, we compare the second and the first period payoffs and we find regions where $\Pi_A^1 \geq \Pi_A^2$, hence we find $\underline{\delta}$. By this comparison we find thresholds $(\underline{\delta}, \overline{\delta})$ and these thresholds are $\delta = 0.57$ and $\delta = 0.69$. For δ's smaller than 0.57, time-consistent agent choose $\theta_1^{*]}$ and ends the game in the first period. For $0.57 < \delta < 0.83$ the game ends in the second period, and agent choose a optimal stream $(\theta_1^{**}, \theta_2^{**})$ that ends game at second period. For $\delta > 0.83$ the game ends in the second period, and agent chooses an optimal stream of investment levels, $(\theta_1^{***}, \theta_2^{***}, \theta_3^{***})$, that fends the game in the third period.

6.2 Sophisticated Agent

(1) To get the optimal investment level in the second period for the sophisticated agent to take the game to last period, the agent maximizes $\delta\beta[s\theta_3 - (\theta_3 - \theta_2)^2] - (\theta_2 - \theta_1)^2$ where $\theta_3 = \theta_2 + s/2$.

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[20]The Matlab codes are available on request.

That is, the agent solves

$max_{\theta_2} \, \delta\beta[s\theta_2 - s^2/2 - s^2/4] - (\theta_2 - \theta_1)^2$ and the optimal θ_2^{***}, which is the optimal investment level in the second period that ends the game in the third period, is obtained as

$\theta_2^{***} = \theta_1^{***} + s\delta\beta/2$

(2) We get the optimal stream of investment levels that will end the game in the second period. We again define $\frac{(2\delta-1)s}{2(1-\delta)} = K$.

Suppose $\theta_1 < K$. Then $\theta_2 = \frac{s\delta}{2(1-\delta)}$. Then, solving

$$max_{\theta_1} \delta\beta[s\theta_2 - (\theta_2 - \theta_1)^2] - \theta_1^2$$

we get, $\frac{\delta\beta\theta_2^{**}}{1+\delta\beta} = \theta_1^{**}$, which implies

$$\theta_1^{**} = \frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)}$$

Moreover for θ_1^* to be smaller than K we need

$$\frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)} < \frac{(2\delta-1)s}{2(1-\delta)} \Rightarrow 1 < 2\delta - \delta\beta + \delta^2\beta$$

Now suppose $\theta_1 \geq K$, and we have $\theta_2 = \theta_1 + s/2$. Solving

$$max_{\theta_1} \delta\beta[s(\theta_1 + \frac{s}{2}) - (\theta_2 - \theta_1)^2] - \theta_1^2$$

we get $\theta_1^{**} = \delta\beta s/2$. Hence, the optimal stream of investment to end the game in the second period is

$$(\theta_1^{**}, \theta_2^{**}) = \begin{cases} \{\frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)}, \frac{\delta s}{2(1-\delta)}\} & 1 < 2\delta - \delta\beta + \delta^2\beta \\ \{\frac{s\delta\beta}{2}, \frac{s(1+\delta\beta)}{2}\} & 1 \geq 2\delta - \delta\beta + \delta^2\beta \end{cases}$$

(3) We get the optimal stream of investment levels that will end the game in the first period. We'll check $\Pi_1^p > \Pi_2^p$ and $\Pi_1^p > \Pi_3^p$.

Suppose $\theta_1 < K$. First condition, $\Pi_1^p > \Pi_2^p$, implies $(1-s)\theta_1 > \delta(1-s)\theta_2$ and in this region we have $\theta_2^{**} = \frac{\delta s}{2(1-\delta)}$ hence θ_1 must satisfy $\theta_1 > \frac{s\delta^2}{2(1-\delta)}$. Second condition $\Pi_1^p > \Pi_3^p$

implies $\delta^2(1-s)(\theta_3) < (1-s)\theta_1$ and in this region we have $\theta_3^{***} = \theta_1 + \frac{s}{2} + \frac{s\delta\beta}{2}$. Hence, we get $\theta_1 < \frac{\delta^2 s(1+\delta\beta)}{2(1-\delta^2)}$. However first condition contradicts with $\theta_1 < K$ since $\frac{(2\delta-1)s}{2(1-\delta)} > \theta_1 > \frac{s\delta^2}{2(1-\delta)}$, which implies $0 > (\delta-1)^2$ which is a contradiction. Hence ending in the first period is not profitable for the principal for the first case.

Thus, suppose $\theta_1 \geq K$. We will check $\Pi_1^p > \Pi_2^p$ and $\Pi_1^p > \Pi_3^p$. First condition, $\Pi_1^p > \Pi_2^p$, implies $(1-s)\theta_1 > \delta(1-s)\theta_2$ and in this region $\theta_2^{**} = \theta_1 + s/2$. Hence θ_1 must satisfy $\theta_1 > \frac{\delta s}{2(1-\delta)}$. Second condition, $\Pi_1^p > \Pi_3^p$ implies $\delta^2(1-s)\theta_3 < (1-s)\theta_1$ and in this region we have $\theta_3^{***} = \theta_1 + \frac{s}{2} + \frac{s\delta\beta}{2}$. Hence $\theta_1 > \frac{\delta s(1+\delta\beta)}{2(1-\delta^2)}$. However, $\frac{\delta s}{2(1-\delta)} > \frac{\delta^2 s}{2(1-\delta)}$, hence $\frac{\delta s}{2(1-\delta)}$ is the minimum investment level that guarantees to get partnership offer when we compare the second and the third period payoffs with the one in first period. Hence the optimal amount of investment in the first period is as follows:

$$
\theta_1^* = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > 2\delta^2 + \delta^3\beta \\[2mm] \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}
$$

(4) Analytically ending game at t=2 gives the following profit by the viewpoint of today:

$$
\delta\beta[s\theta_2 - (\theta_2 - \theta_1)^2] - \theta_1^2
$$

$$
(\theta_2^{**} - \theta_1^{**}) = \begin{cases} \frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)} - \frac{\delta s}{2(1-\delta)} & 1 < 2\delta + \delta^2\beta - \delta\beta \\[2mm] \frac{s\delta\beta}{2} - \frac{s(1+\delta\beta)}{2} & 1 \geq 2\delta + \delta^2\beta - \delta\beta \end{cases}
$$

Or simply,

$$
(\theta_2^{**} - \theta_1^{**}) = \begin{cases} \frac{\delta s}{2(1-\delta)(1+\delta\beta)} & 1 < 2\delta + \delta^2\beta - \delta\beta \\[2mm] s/2 & 1 \geq 2\delta + \delta^2\beta - \delta\beta \end{cases}
$$

For medium values of δ terminating game at t=1 and t=2 gives same investment level $\theta_1 = \theta_2 = \frac{s\delta}{2(1-\delta)}$. As β rises, it becomes optimal to distribite investment between first and second period since cost function is quadratic and cost decreases $(\delta\beta(\frac{\delta s}{2(1-\delta)(1+\delta\beta)})^2)$ as β rises. Moreover, profit $\delta\beta s(\frac{s\delta}{2(1-\delta)})$ is also increases as β rises. Hence, agent become more incline to spread investment between first and second period as β increases.

6.3   Naive Agent

Given initial investment level θ_1^{**}, where

$$\theta_1^{**} = \begin{cases} \frac{s\delta^2}{2(1-\delta)(1+\delta)} & 1 < \delta^2 + \delta \\[2mm] \frac{s\delta}{2} & 1 \geq \delta^2 + \delta \end{cases}$$

the agent calculates the optimal stream that takes the game to the third period.

First case: $1 < \delta^2 + \delta$ which gives $\theta_1^{**} = \frac{s\delta^2}{2(1-\delta)(1+\delta)}$ and solving

$$\max_{\theta_2} \; \delta\beta[s\theta_3 - (\theta_3 - \theta_2)^2] - (\theta_2 - \theta_1)^2$$

where $\theta_3 = \theta_2 + s/2$, hence optimal we get

$$\delta\beta s/2 + \theta_1^{**} = \theta_2^{***}$$

$$s/2 + \theta_2^{***} = \theta_3^{***}$$

that is,

$$\frac{\delta\beta s}{2} + \frac{s\delta^2}{2(1-\delta)(1+\delta)} = \theta_2^{***}$$

$$\frac{\delta\beta s}{2} + \frac{s\delta^2}{2(1-\delta)(1+\delta)} + \frac{s}{2} = \theta_3^{***}$$

Second case: $1 \geq \delta^2 + \delta$ implies $\theta_1^{**} = \frac{s\delta}{2}$. Similarly, we get

$$\frac{\delta\beta}{2} + \frac{s\delta}{2} = \theta_2^{***}$$

$$\frac{\delta\beta s}{2} + \frac{s\delta}{2} + \frac{s}{2} = \theta_3^{***}$$

Given initial investment θ_1^{***}, the optimal θ_2 that ends the game in the second period is equal to $\theta_2^* = \theta_1 + \frac{s}{2}$. Hence, $\theta_2^* = \frac{\delta^2 s}{2} + \frac{s}{2} = \frac{s}{2}(\delta^2 + 1)$. However, we have to check whether or not this optimal stream gets an offer. To see this, note that $\frac{s}{2}(\delta^2 + 1) \geq \frac{s\delta}{2(1-\delta)}$ implies $1 \geq \delta^3 + 2\delta - \delta^2$.

Thus,

$$\theta_{=}^{**} \begin{cases} \frac{s(1+\delta^2)}{2} & 1 \geq \delta^3 + 2\delta - \delta^2 \\[2ex] \frac{s\delta}{2(1-\delta)} & 1 < \delta^3 + 2\delta - \delta^2 \end{cases}$$

6.4 Naive Agent vs Sophisticated Agent

Optimal θ_1 levels for naive agent are as follows:

$$\theta_1^{***} = \frac{\delta^2 s}{2}$$

$$\theta_1^{**} = \begin{cases} \frac{s\delta^2}{2(1-\delta)(1+\delta)} & 1 < \delta^2 + \delta \\[2ex] \frac{s\delta}{2} & 1 \geq \delta^2 + \delta \end{cases}$$

$$\theta_1^{*} = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > \delta^2 + \delta \\[2ex] \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}$$

Optimal θ_1 levels for sophisticated agent are as follows:

$$\theta_1^{***} = \frac{\delta^2 s\beta}{2}$$

$$\theta_1^{**} = \begin{cases} \frac{s\delta^2\beta}{2(1-\delta)(1+\delta\beta)} & 1 < \delta^2\beta - \delta\beta + 2\beta \\[2ex] \frac{s\delta}{2} & 1 \geq \delta^2\beta - \delta\beta + 2\beta \end{cases}$$

$$\theta_1^{*} = \begin{cases} \frac{s}{2} & \frac{1}{2} > \delta \text{ and } 1 > 2\delta^2 + \delta^3\beta \\[2ex] \frac{s\delta}{2(1-\delta)} & \text{otherwise} \end{cases}$$

Since $\beta \leqslant 1$, naive agent's initial θ_1 investment levels are bigger than those of the sophisticated agent.