Auctioning a Discrete Public Good under Incomplete Information*

Murat Yılmaz†

Abstract

We study a dynamic auction mechanism in the context of private provision of a discrete public good under incomplete information. The bidders have private valuations, and the cost of the public good is common knowledge. No bidder is willing to provide the good on her own. We show that a natural application of open ascending auctions in such environments fails dramatically: The probability of provision is zero in any equilibrium. The mechanism effectively auctions off the "right" to be the last one to contribute, but intuition suggests that neither player wishes to be the last one to contribute. Since the player who contributes first has the advantage of being able to free ride on the contributions of the other players, no player wants to "win" the auction.

Keywords : discrete public good, open-ascending auction

JEL **Classification Numbers:** H41, D44, D61, D82

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†Boğaziçi University, Department of Economics, Bebek, Istanbul, 34342. Email: *muraty@boun.edu.tr.* Phone: +90 212 3597656. Fax: +90 212 2872453. Web site: *http://www.econ.boun.edu.tr/yilmaz/YILMAZ.*

1 Introduction

Optimal mechanisms, in a setting where provision of a public good is private, involve complicated transfer schemes. These mechanisms seem implausible as a description of *private* provision of public goods. Ledyard and Palfrey (1999) point out this issue: "[...]There are several directions worth pursuing. One direction is to explore the use of simple mechanisms. The public good mechanisms proposed here involve complicated transfer schemes that can necessitate the use of very large taxes and subsidies." We respond to this problem by considering simple and plausibly private mechanisms which are expected to perform well globally.

More specifically, we consider the use of auctions as a mechanism for private provision of public goods under incomplete information. Auction mechanisms have proven to be very useful when allocating private goods under asymmetric information.[1] They are also very simple and familiar. Thus, it is natural to consider auction mechanisms as a way of modelling private provision of public goods. We study an open ascending auction mechanism for public goods and prove a strong negative result regarding its efficiency properties.

We focus on discrete public goods where the provision decision is only whether or not to produce the good. A street light, public radio fund-raising to finance a certain program, a toll-free bridge are typical examples of a discrete public good. The cost of providing the good is publicly known and each agent has a privately known valuation for the good. We show that there is no provision in a natural adaptation of open ascending auctions. We also provide alternative formats and analyze their probability of provision.

In the open ascending auction mechanism we consider, each bidder, observing the ascending price and whether the other bidder has dropped out or not, drops out at her preferred price and a bidder's contribution is her drop-out price. If the contributions add up to a level at least as big as the cost of the public good, then the good is provided; otherwise, there is no provision and no payments are made.[2] The reason for the probability of provision being zero in any equilibrium in this open ascending auction mechanism is the following: Because of the sequential nature of the contributions, each bidder is eager to be the first one to contribute in order to free ride on the contribution of the other bidder. By committing to a low level of contribution, a bidder can force the other to contribute the rest of the cost. Thus, no bidder is willing to win the auction; that is, no bidder wants to be the last one to drop out. Thus, the bids are too low and as a result there is no provision. This is particularly striking, because although the first bidder to drop out is able to free ride on the other bidder, she also faces the risk that the other bidder may not value the public good high enough for it to be

[1] See Krishna (2002) and Myerson (1981).

[2] Here, we describe the 2-bidder version but we provide a more general result for any number of bidders. Also, with a full refund feature, contributors are not worried about potential negative payoffs. Thus, they tend to contribute more relative to the no refund case.

provided.[3]

Private provision of public goods and its efficiency properties have been vastly studied. In a sequential contribution game under complete information where the amount of the public good to be provided is continuous, the ability of the first mover to credibly commit to a certain level of contribution aggravates the free rider problem (Varian (1994)). On the other hand, under incomplete information, a sequential contribution mechanism may perform better than the simultaneous contribution game, in terms of total expected contributions (Bag and Roy (2011)). In a sequential contribution game where the public good is either provided or not, the outcome is inefficient if the contributions are sunk and there is no commitment device. For the case where the contributions are borne if and only if they cover the cost of the public good, the outcome is efficient (Admati and Perry (1991)). However, if there is imperfect information about individual actions (aggregate contribution is observed but not the individual contributions) and if players can contribute more than once and at any time, then efficiency is achieved under certain conditions (Marx and Matthews (2000)).[4] In a model of private provision of public goods, the set of undominated perfect equilibrium outcomes is identical to the core (Bagnoli and Lipman (1992)).[5] The private provision mechanism for a discrete public good through both contribution game and subscription game (a simultaneous contribution game with full refunds) is studied and some interim inefficiency results have been shown (Barbieri and Malueg, (2008a, 2008b)). Barbieri and Malueg (2010) show that, restricting attention to piecewise-linear equilibrium, the subscription game achieves the outcome of the optimal mechanism for a profit maximizing seller. To study continuous equilibria in the private provision game, tools familiar from auction theory are used (Alboth, Lerner, and Shalev (2001) and Menezes, Monteiro and Temimi (2001)). Auction-like mechanisms are also used to allocate excludable public goods (Deb and Razzolini (1999)). Mechanisms in the spirit of k-th lowest bidder auctions are used in a setting where a group of people must determine which of its members should provide an indivisible public good (Kleindorfer and Sertel (1994)). Schmitz (1997) considers a profit maximising monopolist who provides a discrete and excludable public good to a group of n potential consumers whose valuations are private information, and characterizes the optimal contract.

[3]A natural variation of the open ascending auction described above is where the auction stops once the first bidder drops out and the other bidder simply decides whether to pay the rest of the cost or not. It turns out our main result is still valid: the probability of provision is zero. For the proof, please see Appendix B in the working paper Yılmaz (2010).

[4]The horizon must be long, the players must have similar preferences and they must be patient enough.

[5]For more on efficiency properties of the outcomes of the private provision of public goods, see also Bliss and Nalebuff (1984), Laussel and Palfrey (2002, 2003, 2007), Lu and Quah (2009), Morelli and Vesterlund (2000).

2 The Model

There is a discrete public good, the provision decision of which is binary. There are $N \geq 2$ bidders, A and B. Each bidder i assigns a value x_i to the public good. Each valuation x_i is a realization of a random variable $X_i \in [0,1]$, and each X_i is independently and identically distributed on $[0,1]$ according to the cumulative distribution function $F(\cdot)$ with a full support. The associated density function is $f(\cdot)$. Bidders are risk neutral. Bidder i knows only the realization $x_i \in X_i$ and that the other bidders' valuations are drawn from $F(\cdot)$. The cost of providing the public good is given by a constant c such that $N > c > N-1$, thus, no bidder is willing to provide the public good on her own. And, if at least one bidder does not contribute at all, then there is no chance of provision. Also, there are some combinations of valuations for which it is efficient to provide the good as well as other combinations for which it is not efficient to do so. All of the above is common knowledge to all bidders.

2.1 All-pay open ascending auction

In an open ascending all-pay auction mechanism, the bidders pay their drop-out prices. The payments are contributed towards the provision of the public good, and they are fully refunded if there is no provision.

The rules of the auction are as follows: The auctioneer sets the price at zero and continuously raises it. Each bidder observes the ascending price and drops out at her preferred price. Bidders may drop out at any price, but once they do so, they cannot reenter the auction. The auction ends when there is no active bidder; that is, when all bidders have dropped out. When the auction is over, there are two possibilities: If the sum of the prices at which the bidders have dropped out is at least as big as the cost c, the public good is provided and each bidder pays her own drop out price,[6] and the excess contribution, if any, is kept by the auctioneer. If the sum of the drop-out prices is less than c, then the public good is not provided and no payments are made. If two or more bidders drop out at the same price, the auctioneer selects one of the bidders to be the one to actually drop out by a fair dice.[7] If the public good is provided, bidder i gets a payoff $x_i - b$ where b is her drop-out price. If the good is not provided, each bidder receives zero.

[6] If we let the bidder who drops out first pay nothing, then there is no hope for provision of the public good. Thus, a natural adaptation of open ascending auctions to the context of a discrete public good provision has to have an all-pay feature: the bidder who drops out first pays some amount and a natural candidate is her own bid.

[7] Since we will allow bidding functions that are constant over some range, we need a tie-breaking rule.

3 Equilibrium Analysis

First, we carry out the equilibrium analysis when there are only two bidders. Then, we generalize our result to the $N \geq 2$ case, where we use the result from the two-bidder case.

3.1 $N = 2$ bidders

A strategy for bidder i is a pair denoted (β_{i1}, β_{i2}), where $\beta_{i1} : [0,1] \to \mathbb{R}_+$, and $\beta_{i2} : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$. $\beta_{i1}(x_i)$ is bidder i's drop-out price given her valuation and that the other bidder has not dropped out yet. $\beta_{i2}(x, b_1)$ is her drop-out price given that the other bidder has already dropped out at price b_1. Thus, we require $\beta_{i2}(x, b_1) \geq b_1$.

Our equilibrium concept is Perfect Bayesian Equilibrium, *equilibrium* henceforth. An equilibrium is a strategy profile $((\beta_{A1}, \beta_{A2}), (\beta_{B1}, \beta_{B2}))$ and beliefs $f(\cdot|b_1)$ such that at any stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from the equilibrium strategies and observed actions using Bayes' rule, whenever possible.

The bidder who drops out second, essentially chooses between staying active until the price reaches a level which is enough to cover the rest of the cost or dropping out before the price reaches that level. Thus, in any equilibrium, β_{i2} takes the following form

$$\beta_{i2}(x_i, b_1) = \begin{cases} c - b_1 & \text{if} \quad b_1 < c/2 \text{ and } x_i - (c - b_1) \geq 0 \\ b \in [b_1, c - b_1) & \text{if} \quad b_1 < c/2 \text{ and } x_i - (c - b_1) < 0 \\ b_1 & \text{if} \quad b_1 \geq c/2 \end{cases}$$

By the full-refund assumption, if the first bidder's bid is not enough for the second bidder to be willing to cover the rest of the cost, the second bidder can drop out at any price b less than $c - b_1$, and the public good is not provided. If the first bidder drops out at a price b_1 equal to or higher than $c/2$, then there is provision regardless of the second bidder's drop-out price. Thus, the second bidder drops out immediately at b_1 to minimize her payment.

Given $\beta_{i2}(x_i, b_1)$, for $i = A, B$, above, to describe an equilibrium, we need to specify only $\beta_{i1}(\cdot)$, $i = A, B$, instead of which we write $\beta_i(\cdot)$. If i stands for bidder $A(B)$, then $-i$ denotes bidder $B(A)$. Also, whenever convenient, we use x and y, instead of x_i and x_{-i}, respectively. Let β_i^{-1} denote β_i's inverse. In what follows, we use the terms *contribution*, *drop-out price* and *bid* interchangeably, similarly the terms *contributor* and *bidder*.

Let $\pi^i(x, b, \beta_{-i})$ denote the expected payoff of bidder i from bidding b when her value is x and bidder $-i$ bids according to $\beta_{-i}(\cdot)$. If bidder i is the last one to drop out and the first bidder has dropped out at a price less than $c/2$, then her payoff is zero if she chooses not to pay the rest of the cost and $x - (c - \beta_{-i}(y))$ if she chooses to pay the rest. If bidder i is the last one to drop out and the first bidder has dropped out at a price higher than $c/2$, then she drops out immediately at the first bidder's drop-out price, thus there is provision and her payoff is $x - \beta_{-i}(y)$. If she is the first one to drop out, then she gets a payoff $x - b$ if the other bidder contributes the rest, and zero otherwise.[8] Thus, given the indicator function,

$$I_{A(x,y)} = \begin{cases} 1 & \text{if } x, y \text{ satisfy condition } A(x,y) \\ 0 & \text{otherwise} \end{cases}$$

the payoff for bidder i with value x when she bids b is given by

$$\pi^i(x, b, \beta_{-i}) = \begin{cases} x - \beta_{-i}(y) & \text{if} \quad b \geq \beta_{-i}(y) \geq c/2 \\ (x - (c - \beta_{-i}(y)))I_{x-(c-\beta_{-i}(y))\geq 0} & \text{if} \quad b > \beta_{-i}(y) \text{ and } \beta_{-i}(y) < c/2 \\ x - b & \text{if} \quad c/2 \leq b < \beta_{-i}(y) \\ x - b & \text{if} \quad b < \beta_{-i}(y),\ y + b > c \text{ and } b < c/2 \\ 0 & \text{if} \quad b < \beta_{-i}(y),\ y + b < c \text{ and } b < c/2 \\ 1/2(x - (c - \beta_{-i}(y)))I_{x-(c-\beta_{-i}(y))\geq 0} & \\ \quad +1/2(x - b)I_{y+\beta_i(x)\geq c} & \text{if} \quad b = \beta_{-i}(y) < c/2 \end{cases}$$

Let μ_{β_i} be the probability measure over $\{\beta_i(x) : x \in [c-1, 1]\}$. Define $\overline{\beta}_i = \mathbf{max}$ $\mathbf{supp}(\mu_{\beta_\mathbf{i}})$, where *supp*$(\mu_{\beta_i})$ is the support of μ_{β_i}.

In an equilibrium $(\beta_A(\cdot), \beta_B(\cdot))$, with $\overline{\beta}_i \leq c-1$ for some $i \in \{A, B\}$, there is no provision. To see this, if a bidder is bidding above $c - 1$ when her valuation is less than $c - 1$, then she must be the second to drop out almost everywhere. Otherwise, she receives a negative payoff when she is the first one to drop out, and zero when she is the second to drop out. Thus, she profitably deviates to a bid below $c - 1$ and gets zero payoff. If $\overline{\beta}_i \leq c - 1$, then bidder $-i$, if she is the second to drop out, will not pay the rest of the cost since valuations are at most 1. If bidder $-i$ is the first one to drop out, then her bid must be less than $c - 1$. Thus, bidder i will not pay the rest of the cost. Thus, there is no provision of the public good and both bidders get zero payoff. We call such an equilibrium an **equilibrium with zero probability of provision**.

Definition 1 *An* ***equilibrium with zero probability of provision*** *in a 2-bidder game is a strategy pair,* $(\beta_A(\cdot), \beta_B(\cdot))$*, and beliefs such that strategies are sequentially rational and beliefs are consistent, where* $\overline{\beta}_i \leq c - 1$ *for some* $i \in \{A, B\}$*.*

An ***equilibrium with positive probability of provision*** *in a 2-bidder game is a strategy pair,* $(\beta_A(\cdot), \beta_B(\cdot))$*, and beliefs such that strategies are sequentially rational and beliefs are consistent, where* $\overline{\beta}_i > c - 1$ *for each* $i \in \{A, B\}$*.*

[8] If bidder i is the first to drop out with a drop-out price b less than $c/2$, then it is irrelevant whether the other bidder has a value y such that $y + b = c$. This is because such an event has probability zero. Thus, we omit the case $y + b = c$.

An equilibrium with zero probability of provision always exists. Any $(\beta_A(\cdot), \beta_B(\cdot))$ with $\beta_i(x) \leq c-1$ for all x, for both $i = A, B$ is an equilibrium with zero probability of provision. To see this, first note that both bidders get zero payoff since there is no provision of the public good. Given that bidder i uses a strategy $\beta_i(x)$ with $\beta_i(x) \leq c-1$ for all x, bidder $-i$ has no profitable deviation. This is because the good will never be provided even if bidder $-i$ drops out at a price higher than $c-1$ for some positive measure set of values.

We are interested in the set of equilibria with positive probability of provision and whether there exists one. Note that in an equilibrium with positive probability of provision, the probability of provision is strictly positive since $\overline{\beta}_i > c-1$ for each $i \in \{A, B\}$.

Given that the other bidder has not dropped out yet, bidder i drops out before the price exceeds half of the cost.

Lemma 1 *In an equilibrium with positive probability of provision,* $(\beta_A(\cdot), \beta_B(\cdot))$*,* $\beta_i(x) \leq c/2$ *for each* $x \in [0,1]$*, for* $i = A, B$*.*

Proof. If no one has dropped out yet when the price is $c/2$, there will be provision for sure where both bidders pay at least $c/2$. In that case, it is optimal to drop out immediately at $c/2$, because otherwise, bidders pay more than necessary. Thus, bidders wait at most until the price reaches $c/2$. ■

But, the price at which a bidder drops out given that the other bidder has already dropped out can be above $c/2$.

First, we focus on equilibrium bidding functions which are non-decreasing. The following lemma below shows that in an equilibrium with positive probability of provision with non-decreasing bidding functions both bidders receive a positive expected payoff for large enough valuations.

Lemma 2 *In an equilibrium with positive probability of provision,* $(\beta_A(\cdot), \beta_B(\cdot))$*, with non-decreasing bidding functions,* $\pi^i(x, \beta_i(x), \beta_{-i}) > 0$ *for each* $x \in (c-1, 1]$*, for* $i = A, B$*.*

Proof. Suppose $(\beta_A(\cdot), \beta_B(\cdot))$ is an equilibrium with positive probability of provision. By definition $\overline{\beta}_i > c-1$ for $i = A, B$. Suppose that bidder i with a value $x \in (c-1, 1]$ receives a zero expected payoff, $\pi^i(x, \beta_i(x), \beta_{-i}) = 0$. But then bidder i could deviate to a bid $c-1+\varepsilon$, for some small $\varepsilon > 0$ such that $c-1+\varepsilon < \min\{x, \overline{\beta}_{-i}\}$ and get a positive expected payoff. To see this, note that the fact that this bid is strictly below $\overline{\beta}_{-i}$ and the fact that β_{-i} is non-decreasing imply that there is a positive measure set of values of bidder $-i$, say $[1-\delta, 1]$, such that bidder $-i$ drops out second when bidder i drops out at $c-1+\varepsilon$. Since $-i$ would pay the rest of the cost whenever her valuation is in $(1-\varepsilon, 1]$, the public good is provided. Thus bidder i's payoff is strictly positive when bidder $-i$'s valuation is in $(\max\{1-\delta, 1-\varepsilon\}, 1]$, a set with strictly positive measure. Since an equilibrium expected payoff can never be strictly smaller than the expected profit from a deviation, the result follows. ■

The next Lemma states that, in the relevant range, no bidder bids more than her valuation for the public good in an equilibrium with positive probability of provision with non-decreasing bidding functions.

Lemma 3 *In an equilibrium with positive probability of provision, $(\beta_A(\cdot), \beta_B(\cdot))$, with non-decreasing bidding functions, $\beta_i(x) \leq x$ for each $x \in (c-1, 1]$, for $i = A, B$.*

Proof. Suppose $\beta_i(\widehat{x}) > \widehat{x}$ for some $\widehat{x} \in (c-1, 1]$. By Lemma 1, $c/2 \geq \beta_i(\widehat{x}) > \widehat{x}$. Suppose the price has just reached $\widehat{x}$. If bidder $-i$ drops out at this instant, then bidder i's payoff is zero, when her valuation is $\widehat{x}$. To see this, note that bidder $-i$ pays $\widehat{x}$ and bidder i pays at most $\widehat{x}$, making at most $2\widehat{x}$ in total, and $2\widehat{x} < c$. Thus, the public good is not provided. If bidder $-i$ does not drop out at $\widehat{x}$, then bidder i's expected payoff is strictly negative unless the probability of provision is zero. To see this, note that if the public good is provided, then bidder i pays at least the lower of the two drop-out prices. If she is the first to drop out, she pays her bid which is above $\widehat{x}$. If she is the second to drop out she pays the rest of the cost which is at least $c/2$, thus, strictly greater than $\widehat{x}$. Thus, when the price reaches $\widehat{x}$, bidder i keeps being active only if the probability of provision is zero conditional on price reaching $\widehat{x}$. Thus, her expected payoff is zero. By Lemma 2, this cannot be the case in an equilibrium with positive probability of provision with non-decreasing bidding functions. ■

When $\beta_i(x)$ is non-decreasing almost everywhere over the range $(c-1, 1]$, let x^{β_i} be the valuation at the intersection of $\beta_i(x)$ with the line $c = b + x$. More formally, define x^{β_i} to be the value such that $\beta_i(x) \geq c - x^{\beta_i}$ almost everywhere over the range $(x^{\beta_i}, 1]$, and $\beta_i(x) \leq c - x^{\beta_i}$ almost everywhere over the range $(c-1, x^{\beta_i})$.[9]

Let $b^{\beta_i} = c - x^{\beta_i}$. Interpretation of b^{β_i} is as follows. Whenever price reaches b^{β_i}, the other bidder, $-i$, knows that the if she drops out at that price, bidder i will be willing to pay the rest of the cost. Thus, for bidder $-i$, there is no point waiting further and letting the price go over b^{β_i}.

Lemma 4 *In an equilibrium with positive probability of provision, $(\beta_A(\cdot), \beta_B(\cdot))$ with bidding functions non-decreasing almost everywhere over the range $(c-1, 1]$, $\beta_i(x) \leq b^{\beta_{-i}}$, for almost all $x \in (c-1, 1]$, for $i = A, B$.*

Proof. See the Appendix. ■

[9]If $\beta_i(x) \leq c-1$ for almost all x, then $x^{\beta_i} = 1$. If $\beta_i(x) \geq c$ for almost all x, then $x^{\beta_i} = 0$. In any other case, there exists a $x^{\beta_i} \in (0, 1)$. To see this, first note that if there exist a value x' such that $\beta_i(x') = c - x'$, then there is exactly one such x', since β_i is non-decreasing. Thus, $x^{\beta_i} = x'$. If there is no such x', then there exists a unique value x'' such that $\beta_i(x) > c - x''$ for all $x > x''$, and $\beta_i(x) < c - x''$ for all $x < x''$, since β_i is non-decreasing. Thus, $x^{\beta_i} = x''$.

The intuition for Lemma 4 is as follows: As noted above, when the price reaches $b^{\beta_{-i}}$, player i should drop out immediately. Thus, her bidding function should not exceed $b^{\beta_{-i}}$ for any of her valuation. More specifically, if player i bids more than $b^{\beta_{-i}}$, then she can increase her expected payoff by cutting her bid. In Figure 1a below, the bidder i, bidding b, has a valuation x that is less than $c-b$. Cutting her bid, bidder i, expands the range where she gets a positive payoff, and shrinks the range where she gets zero. In Figure 1b below, however, the bidder i has a valuation that is more than $c-b$. In this case, she can increase her expected payoff by cutting her bid, which expands the range where she gets a payoff $x-b$ and shrinks the range where she receives $x-c+\beta_i(y)$ which is less than $x-b$ by Lemma 1. The range where she receives zero is not affected by cutting her bid.

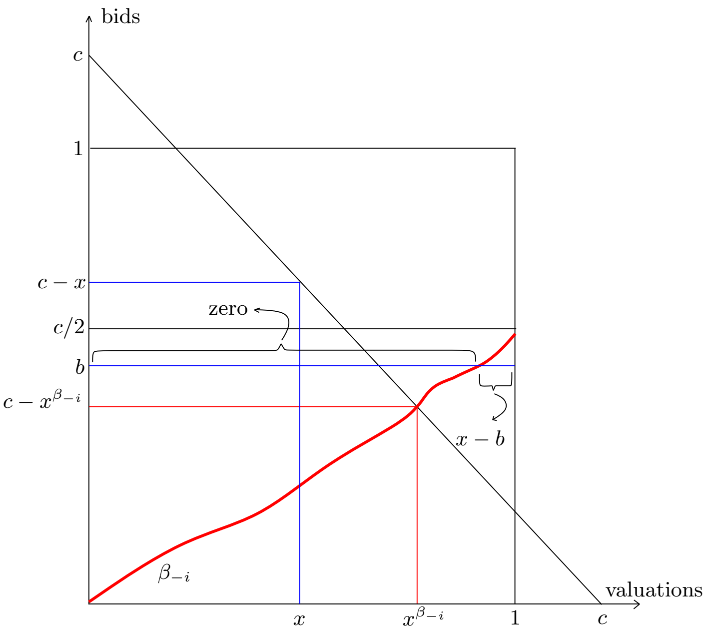

Figure 1a

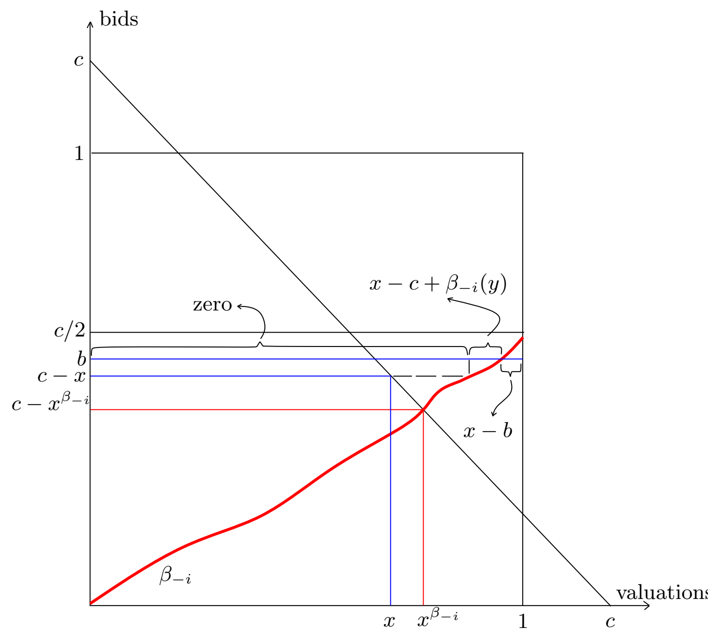

Figure 1b

Lemma 5 *There exists no equilibrium with positive probability of provision,* $(\beta_A(\cdot), \beta_B(\cdot))$, *with bidding functions that are non-decreasing almost everywhere over the range* $(c-1, 1]$.

Proof. See the Appendix. ■

The intuition behind Lemma 5 is depicted in Figure 2. Lemma 4 implies that in an equilibrium with positive probability of provision with non-decreasing bidding functions, $(\beta_A(\cdot), \beta_B(\cdot))$, for all $x > x^{\beta_{-i}}$, $\beta_i(x) = c - x^{\beta_{-i}} = b^{\beta_{-i}}$ for $i = A, B$. Thus, $b^{\beta_A} = b^{\beta_B}$. Then, bidder i of any type who bids b^{β_i} deviates by cutting her bid by a small enough amount. By bidding b^{β_i}, bidder i is the first one to drop out with probability 1/2 for the values $x \in [x^{\beta_{-i}}, 1]$ of bidder $-i$. By deviating to $b^{\beta_i} - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, bidder i becomes the first one to drop out with probability 1 for all $x \in [x^{\beta_{-i}}, 1]$. In case of provision, the first bidder to drop out pays less than $c/2$, while the other bidder pays more than $c/2$, and over the range $[x^{\beta_i}, 1]$ bidder i decreases the probability of provision by only a small amount by cutting her bid by ε. Thus, the loss can be made arbitrarily small by taking ε sufficiently small while the gain has a lower bound strictly greater than zero for all ε.

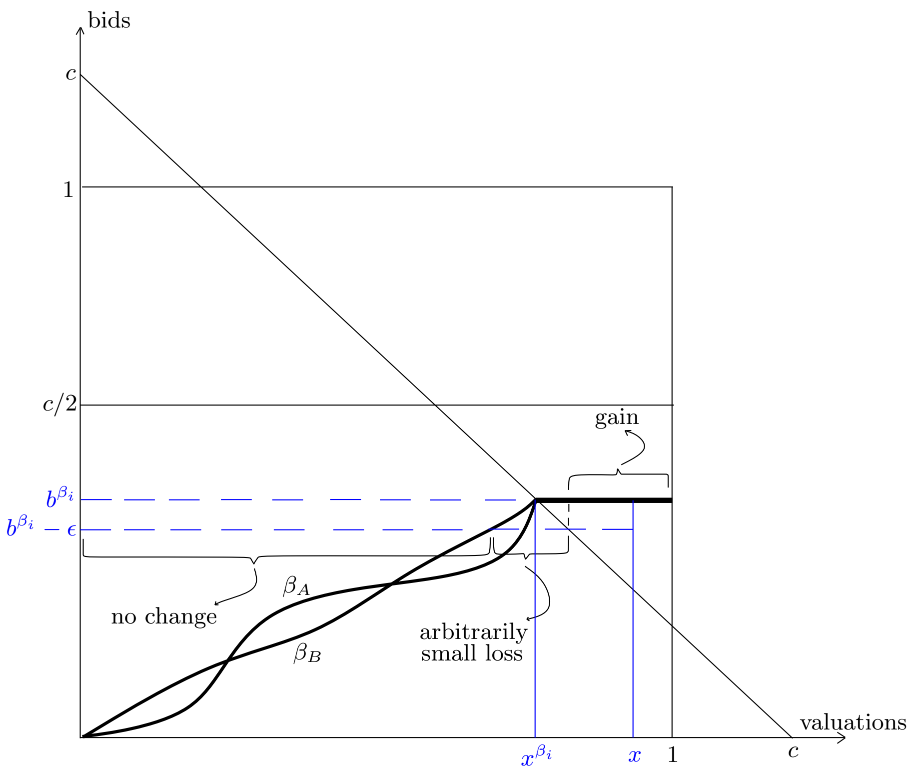

Figure 2

Lemma 6 *Consider an equilibrium with positive probability of provision, $(\beta_A(\cdot), \beta_B(\cdot))$. For any x_1, $x_2 \in (c-1, 1]$ with $x_1 < x_2$, if $\beta_i(x_1) \leq x_1$ and $\beta_i(x_2) \leq x_2$, then $\beta_i(x_1) \leq \beta_i(x_2)$, for $i = A, B$.*

Proof. See the Appendix. ■

The next Lemma shows that an equilibrium with positive probability of provision does not exist.

Lemma 7 *An equilibrium with positive probability of provision does not exist.*

Proof. Suppose there exists an equilibrium with positive probability of provision, $(\beta_A(\cdot), \beta_B(\cdot))$. By definition, $\overline{\beta}_i > c - 1$ for each $i \in \{A, B\}$, thus the probability of provision is strictly positive. If the probability of provision is strictly positive when bidder i has a valuation x, then the probability of provision must be strictly positive when bidder i has a valuation $x' > x$. Otherwise, bidder i's payoff is zero when she has valuation x', but she can deviate to the bid of type x and have a strictly positive expected payoff. Now define x_i^* to be the infimum over the set of values of bidder i such that the probability of provision in this equilibrium is positive.[10] Then, the probability of provision is strictly positive for all $x \in (x_i^*, 1]$. Note that $x_i^* \geq c - 1$, since in any equilibrium the probability of provision is zero when $x \leq c - 1$. Since we assumed that the equilibrium we fixed has a strictly positive probability of provision, $x_i^* < 1$. Note that in the proof of Lemma 3 we showed that if for bidder i, $\beta_i(x) > x$ for some $x \in (c - 1, 1]$, then the bidder i's expected payoff is zero when her valuation is x. Since the expected payoff is strictly positive for any $x \in (x_i^*, 1]$, we conclude that for any $x \in (x_i^*, 1]$, $\beta_i(x) \leq x$.[11] By Lemma 6, $\beta_i(x)$ is non-decreasing over the range $(x_i^*, 1]$. Since the probability of provision is zero for all $x \in [0, x_i^*]$, this cannot be an equilibrium with positive probability of provision by a similar argument used in the proofs of Lemma 4 and Lemma 5. Thus, the probability of provision cannot be strictly positive, which is a contradiction. Thus, an equilibrium with positive probability of provision does not exist. ■

Therefore, the set of equilibria consists of only equilibria with zero probability of provision. Thus, we immediately get our main result.

Theorem 1 *In the all-pay open ascending auction with $N = 2$, the probability of provision is zero in any equilibrium.*

The intuition for this result is as follows: First observe that in a sequential contribution game with exogenous order, being the first one to contribute is advantageous since the first player to contribute, by committing to a low level of contribution, can free ride, in expectation,

[10] Note that x_i^* depends on the equilibrium. To save space we write x_i^* instead of $x_i^*(\beta_A(\cdot), \beta_B(\cdot))$.

[11] In Lemma 3's proof we first show that if $\beta_i(x) > x$ for some $x \in (c - 1, 1]$ then the expected payoff is zero, that is, if the expected payoff is strictly positive then $\beta_i(x) \leq x$ for all $x \in (c - 1, 1]$. Since strictly positive expected payoff is ensured by having non-decreasing bidding functions, the proof was completed. Here, however, we do not impose bidding functions to be non-decreasing. Since for any $x \in (x^*, 1]$, the expected payoff is strictly positive, it follows that $\beta_i(x) \leq x$.

on the other player's contribution. In the all-pay auction game, being the first to drop out is essentially being the first contributor and hence advantageous. Thus, for a bidder who is the second to drop out, it will be profitable to undercut the other's bid to be the first one to contribute. This incentive to deviate disappears only when almost all bids are less than $c-1$ for values that are at least $c-1$, that is, when the equilibrium has zero probability of provision.

3.2 $N > 2$ bidders

When there are $N > 2$ bidders, a bidder's strategy consists of N bidding function and only one of them is trivially obtained, the one when every other bidder has dropped out. Thus, for a bidder i, we have to solve for $N-1$ bidding functions, β_i^k, where $k = 0, , N-2$, and β_i^k is the bidding function when exactly k bidders have already dropped out. Although this creates a complexity in the analysis, still, Theorem 1 can be generalized to the case with $N > 2$ bidders, whenever $c > N-1$, that is, the cost is such that $N-1$ bidders is not enough to provide the public good, where each bidder's valuation is drawn from $[0,1]$. First, we prove the following lemma.

Lemma 8 *Suppose the cost of the public good is c such that $N > c > N-1$. Then, if the first drop out price is b_1 with $c/N > b_1 > c-N+1$, then $n-1 > c' > N-2$ where $c' = c-b_1$.*

Proof. Let $b_1 = c-N+1$. Then, $c' = c-b_1 = N-1$. Thus, if $b_1 > c-N+1$, then $c' < N-1$. If $b_1 = c/N$, then $c' = c-(c/N) = \frac{N-1}{N}c > \frac{N-1}{N}(N-1) > N-2$. The last strict inequality follows from $N^2-2N+1 > N(N-2) = N^2-2N$. ■

Now, we prove the generalized version of Theorem 1.

Theorem 2 *In the all-pay open ascending auction with $N \geq 2$ and $N > c > N-1$, the probability of provision is zero in any equilibrium.*

Proof. Suppose we have an equilibrium with positive probability of provision. Then, the first drop out price, b_1, in the equilibrium must be such that $b_1 > c-N+1$, otherwise there is no chance of provision since $c-(c-N+1-\epsilon) = N-1+\epsilon > N-1$, for any $\epsilon > 0$. Thus, the residual cost is too much for the rest of the $N-1$ bidders to pay it. Also, note that Lemma 1 applies here as well. Thus, when no one has dropped out yet, the drop out price cannot be larger than c/N. Therefore, $c/N > b_1 > c-N+1$. Then, by Lemma 8, the residual cost is c' such that $N-1 > c' > N-2$. Therefore, we have reduced game in which the cost is c' with $N-1 > c' > N-2$ and the number of players is $N-1$. The price is now starting from b_1 but since the cost is normalized to $c' = c-b_1$, this reduced game is as if its starting price is zero. Thus, in this reduced game with $N-1$ bidders, we can apply Lemma 8 again, and reduce it further to a new game with $N-2$ bidders. Repeatedly applying Lemma 8 we reduce the whole game to a game with 2 players where the residual cost is between 2 and 1, for which Theorem 1 applies. Thus, there is no equilibrium with positive probability of provision. ■

The theorem above is valid when the cost of the public good is sufficiently large, that is, when $c > N-1$. When the cost is smaller, say $c < N-1$, however, there may be equilibria with positive probability of provision. To see this, let's look at the case where $N=2$ and suppose $0 < c < 1$. The profitable deviations in Lemma 4 and Lemma 5 would still be valid. However, now the strategies $(\beta_A(\cdot), \beta_B(\cdot))$ with $\overline{\beta_i} = 0$, for $i = A, B$,[12] would not be an equilibrium since for some values drawn from $[0,1]$, it is a best response for one of the players to actually provide the good on her own. Thus, the probability of provision is not necessarily zero in any equilibrium. This is not surprising though, since we are considering a case with a lower cost of the public good, low enough that one bidder (when $N=2$) can actually provide the good on her own if she draws a high enough valuation.

4 Alternative Mechanisms

4.1 Entry stage

Suppose there is an entry stage where the auctioneer announces the starting price, r, of the open ascending auction. Observing the starting price, the bidders decide to enter the auction or not. Then, those who decide to enter, participate in the open ascending auction described in section 2.1, with a starting price $r > c-1$.[13] The equilibrium analysis in section 3 still applies and the bidders drop out immediately at price r. Then, one of the bidders is randomly picked and the other bidder simply waits until the price reaches $c-r$ provided her valuation is at least $c-r$, otherwise she drops out at some price before it hits $c-r$. Thus, given that both bidders enter the auction, the probability of provision is $Prob(x \geq c-r) = 1-Prob(x < c-r) = 1-F(c-r)$. The probability of a bidder entering the auction is $Prob(x \geq r)$. This is because, if a bidder with a valuation $x < r$ enters the auction, in the equilibrium she will immediately drop out at price r, and with a 1/2 probability she will be picked to be the first one to drop out, and the other bidder will be kept active. If the other bidder has a valuation higher than $c-r$, then there will be provision in which the bidder with valuation $x < r$ pays r and hence gets a negative payoff. Also, a bidder with a valuation $x \geq r$ receives a non-negative payoff in the auction if she enters. Moreover, the provision probability is zero if exactly one bidder or no bidder enters the auction. For a

[12] $\overline{\beta_i} < c-1$ translates into $\overline{\beta_i} = 0$ when $c < 1$.

[13] If $r \leq c-1$, then in the equilibrium the bidders still drop out before price hits $c-1$, thus the probability of provision is still zero.

positive probability of provision, both bidders must enter. This is because highest possible valuation is 1 and the cost is $c > 1$, thus one bidder on her own cannot provide the good.

Thus, we get the probability of provision to be $[Prob(x \geq r)]^2[1 - F(c-r)] = [1 - F(r)]^2[1 - F(c-r)]$, where $[Prob(x \geq r)]^2$ is the probability of both bidders entering the auction, and $[1-F(c-r)]$ is the probability of provision conditional on both bidders entering the auction. Note that, if $r = c-1$, then this probability is zero. If $r > c-1$ then it is strictly positive. Thus, when there is an entry stage where a high enough starting price is announced and the bidders first decide whether to enter or not, the probability of provision is **strictly positive**.

The optimal starting price r^* satisfies the following necessary condition: $\frac{1-F(c-r^*)}{f(c-r^*)} = \frac{1-F(r^*)}{2f(r^*)}$. For instance, with the uniform distribution over $[0, 1]$, the optimal starting price is $r^* = (2c-1)/3$ where $c/2 > r^* > c-1$. Note that $v(c-r) = (c-r) - \frac{1-F(c-r)}{f(c-r)}$ is known as the virtual valuation of a bidder with valuation $c - r$. Also, $v(r) = r - \frac{1-F(r)}{2f(r)}$ is the virtual valuation of a bidder with valuation r. Thus, at the optimal r^*, we have $(c-r^*) - v(c-r^*) = r^* - v(r^*)$.

4.2 Auctioning off "the right to contribute first"

An alternative way to interpret Theorem 1 is that the bidders want to be the first bidder to drop out in order to leave a bigger portion of the cost to the other bidder. Bidders have an incentive to (almost) free ride, in expected terms, on the contribution of the other bidder by dropping out first at a low level. This suggests that it is valuable for the bidders to be the first to contribute. Thus, one alternative mechanism is to first auction off "the right to contribute first" in the first stage, and then carry out a sequential contribution game in the second stage where the order of contributions is determined in the first stage: the winner of the auction gets to be the first one to contribute in the sequential contribution game.

We take the widely-studied sequential contribution mechanism with exogenous order as our benchmark mechanism. We show that for non-decreasing symmetric equilibria, the mechanism where the right to contribute first is auctioned off weakly outperforms the benchmark mechanism, in terms of the probability of provision, under some conditions.

The rules of the mechanism are as follows. In the first stage the right to contribute first is auctioned off through a second-price sealed bid auction. Each bidder simultaneously bids. The bidder with the highest bid wins and pays the other bidder's bid, b_1.[14] The payment made by the winner is used towards the provision of the public good. In case there is a tie in the auction, the winner is picked by a fair coin toss. In the second stage, the bidders play a sequential contribution game. First, the winner of the auction contributes an amount, k_1, on

[14] The winner observes b_1.

top of what he has already paid in the auction. So her total contribution is $b_1 + k_1$. Then the loser of the auction contributes an amount, k_2, after observing the total contribution made by the winner, $b_1 + k_1$. If the total contribution, $b_1 + k_1 + k_2$, is high enough to cover the cost, c, then the public good is provided. Otherwise, the good is not provided and no payments are made; that is, all payments and contributions are fully refunded. If the public good is provided, then a bidder gets a payoff of $x - b$ where x is her valuation for the public good and b is her total contribution, which is the sum of the contribution she made in the second stage and the price she paid in the auction in case she is the winner.

In what follows, we will call the mechanism described above the **endogenous order** mechanism and the sequential contribution mechanism with exogenous order the **random order** mechanism.

A strategy in the endogenous order mechanism for bidder i is given by $(\beta_i, \kappa_{i1}, \kappa_{i2})$, where $\beta_i : [0,1] \to \mathbb{R}_+$, and $\kappa_{i1} : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. and $\kappa_{i2} : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. $\beta_i(x)$ is bidder i's bidding strategy in the auction. Denote the price paid by the winner of the auction by b_1, and the bid of the winner by b_2, and the contribution made by the winner in the sequential contribution game by k_1. Then $\kappa_{i1}(x, b_1, b_2)$ and $\kappa_{i2}(x, b_1, b_2, k_1)$ are the contributions of bidder i, if she wins the auction and if she loses it, respectively.

An equilibrium is a strategy profile $((\beta_A, \kappa_{A1}, \kappa_{A2}), (\beta_B, \kappa_{B1}, \kappa_{B2}))$ and beliefs $f(\cdot|b_1)$ such that at any stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes' rule whenever possible.[15]

In the second stage, the loser of the auction will simply decide to pay the rest of the cost or not, given his valuation for the public good and the total contribution of the winner. This strategy is the same for both bidders, so we can drop the i subscript. Hence $\kappa_2(x, b_1, b_2, k_1)$ is given by

$$\kappa_2(x, b_1, b_2, k_1) = \begin{cases} c - b_1 - k_1 & \text{if } x - (c - b_1 - k_1) \geq 0 \\ k \in [0, c - b_1 - k_1) & \text{otherwise} \end{cases}$$

$\kappa_{i1}(x, b_1, b_2)$ solves the following maximization problem

$$\max_k (x - k - b_1)[1 - F(c - b_1 - k|b_1)]$$

where $F(\cdot|b_1)$ is the cumulative distribution function associated with the posterior beliefs $f(\cdot|b_1)$. We impose the following assumption on the density function in order to ensure that the objective function in the above maximization problem is strictly concave on the equilibrium path.

[15]The equilibrium concept is Perfect Bayesian Equilibrium as in the previous sections.

Assumption $\frac{2f(x)}{|f'(x)|} > 2 - c$ for all $x \in [0,1]$.

An equilibrium is given by $((\beta_A, \kappa_{A1}, \kappa_2), (\beta_B, \kappa_{B1}, \kappa_2))$ where κ_{A1}, κ_{B1} and κ_2 are characterized above.

The random order mechanism has a unique equilibrium in which both players adopt the strategy $(\tau_1(x), \tau_2(x, k_1))$, where $\tau_1 : [0,1] \to \mathbb{R}_+$ and $\tau_2 : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ are the contributions of the first and second contributor given that the first contributor has already contributed k_1, respectively. By the same argument used for $\kappa_{i2}(x, b_1, k_1)$ above, $\tau_2(\cdot,\cdot)$ is given by

$$\tau_2(x, k_1) = \begin{cases} c - k_1 & \text{if } x - (c - k_1) \geq 0 \\ k \in [0, c - k_1) & \text{otherwise} \end{cases}$$

$\tau_1(x)$ solves the following maximization problem

$$\max_{\tau} (x - \tau) \Pr(y \geq c - \tau)$$

By the assumption above, the objective function is strictly concave and the solution is

$$\tau_1(x) = x - \frac{1 - F(c - \tau_1(x))}{f(c - \tau_1(x))}$$

where $\tau_1(x) \geq 0$.[16] Denote the probability of provision generated in the equilibrium of the random order mechanism by p^*. The following proposition states that the random order mechanism cannot generate a strictly higher probability of provision than the endogenous order mechanism does, for a set of probability distributions.

Proposition 1 *Assume that* $\frac{2f}{|f'|} > 2 - c > \frac{f}{|f'|}$ *and* $f' < 0$*. Then, for any non-decreasing and symmetric equilibrium of the endogenous order mechanism, the probability of provision is at least* p^**.*

Proof. See the Appendix. ■

4.3 Open descending auction

There are other auction formats one can consider. The widely studied contribution game and subscription game (contribution game with full-refunds) can both be thought of as a sealed bid auction, which is a static auction format. Among the well known dynamic auction formats, open ascending auction is already studied above. Here, we discuss an open descending auction version of the provision mechanism. The price starts at a reasonably high level (at least from

[16] Otherwise, the left hand side of $\tau_1(x) = x - \frac{1-F(c-\tau_1(x))}{f(c-\tau_1(x))}$ is negative while the right hand side is $x \geq 0$.

1 which is the highest possible valuation) and then continuously drops. Each bidder observes the descending price and drops out at her preferred price. Bidders may drop out at any price, but once they do so, they cannot reenter the auction. If the bidders drop out at the same price, the auctioneer selects one of the bidders to be the first one to drop out by a fair coin toss. The auction ends when both bidders drop out and each bidder's drop out price is her (potential) contribution. As in the open ascending auction mechanism, here it is again straightforward what the second bidder does in the equilibrium, once the first bidder drops out.

It is clear that a bidder with a valuation $x > c/2$ will never wait the price to drop below $c/2$, given that the other bidder has not dropped out yet. This is because, if no one dropped out yet at price $c/2$, not dropping out and letting the price go below $c/2$ kills any chance of provision since the drop out prices are the contributions. Thus, whenever one bidder has a valuation larger than $c/2$, that bidder will drop out either at $c/2$ or before the price hits $c/2$ and leaving a residual cost at most $c/2$. Thus, the probability of provision is **strictly positive**.

4.4 War of Attrition

It is also interesting to consider a war of attrition type of mechanism. Krishna and Morgan (1997) compare the performance of all-pay auctions and the war of attrition in terms of their expected revenue to the seller, under affiliated signals. They show that the war of attrition generates a larger expected revenue than the all-pay auction.

In the war of attrition version of the private provision mechanism, when the first bidder drops out, the game ends and the price the first bidder drops out is paid by both bidders. Thus, if the first drop out price is less than $c/2$, then there is no provision, otherwise there is provision. Note that, if a bidder has a valuation higher than $c/2$, the bidder drops out at price $c/2$ (there is no point letting the price go over $c/2$). And, if the valuation is less than $c/2$, drop out price is anything strictly less than $c/2$ (note that when there is no provision bidders are fully refunded). Thus, if at least one of the bidders has a valuation less than $c/2$, there is no provision. There is provision only if both players have valuations at least $c/2$. Therefore, the probability of provision is the probability of both bidders having valuations at least as big as $c/2$: $[Prob(x \geq c/2)]^2 = [1 - F(c/2)]^2$, which is **strictly positive.**

Comparing this probability of provision to that of the open descending auction mechanism, it turns out that open descending auction has a probability of provision which is at least as big as the probability of provision in the war of attrition. This is because, in the open descending version, whenever both bidders have valuations at least as big as $c/2$, there is provision. Also, for some cases where one bidder has a valuation less than $c/2$ and the other has higher than $c/2$, there is some chance of provision if the bidder with valuation larger than $c/2$ drops out before the price hits $c/2$. However, with war of attrition version, the provision is possible for only those cases where both valuations are at least $c/2$.

5 Conclusion

We explored the private provision of a discrete public good using auction mechanisms. A natural open ascending auction severely failed: in any equilibrium, there is no provision. No bidder wants to *win* the auction, due to the urge to free-ride on the contribution of the other bidder. Thus, the drop-out prices turn out to be very low, leading to no provision at all.

We considered a number of alternative formats as well, and discussed their probability of provision. When there is a sufficiently large starting price, the probability of provision is no longer zero. We also considered auctioning off the "right to contribute first". We showed that, unlike open ascending version, open descending version of the mechanism has positive probability of provision, which is higher than the probability of provision in the war of attrition type of mechanism. There are three points to make from these observations. First, like in the private good auctions, an entry fee improves the revenue, that is, increasing the total contributions, thus improving the efficiency. Second, the open ascending and open descending formats differ drastically in terms of their efficiency performances, which is unlike the result in the private good auctions. Finally, first mover advantage is present in this setting, and thus, first auctioning off the "right to contribute first" and then letting them to contribute sequentially improves the performance of the random order sequential contribution mechanism.

6 Appendix

Proof of Lemma 4. Consider the values x of bidder i such that $x < y^{\beta_{-i}} = c - b^{\beta_{-i}}$. That is, $c - x > b^{\beta_{-i}}$. Suppose that $\beta_{-i}(.)$ is strictly increasing. (We will consider the values $x > y^{\beta_{-i}}$ and the case where $\beta_{-i}(.)$ is weakly increasing later.) Suppose bidder i bids an amount $b \geq b^{\beta_{-i}}$ when her value is x. There are two cases, $c - x > b \geq b^{\beta_{-i}}$ and $b \geq c - x$.

(i) $c - x > b \geq b^{\beta_{-i}}$: For any y such that $y < \beta_{-i}^{-1}(b)$, bidder $-i$ drops out first and her drop-out price is less than $c - x$. Thus, for such values of bidder $-i$, there is no provision. For values y such that $y > \beta_{-i}^{-1}(b)$, bidder i drops out first and her drop out price, b, is bigger than $c - y$. Thus, there is provision and bidder i gets $x - b$ for such values of bidder $-i$. So, the payoff for bidder i is

$$\pi^i(x, b, \beta_{-i}) = \int_{\beta_{-i}^{-1}(b)}^{1} [x - b] f(y) dy = (x - b)[1 - F(\beta_{-i}^{-1}(b))].$$

As b decreases, both $(x-b)$ and $[1-F(\beta_{-i}^{-1}(b))]$ increase. Thus, in the range $c-x > b \geq b^{\beta_{-i}}$, optimal bid is $b^{\beta_{-i}}$.

(ii) $b \geq c-x > b^{\beta_{-i}}$: In this case, for values y, such that $y < \beta_{-i}^{-1}(c-x)$ there is no provision since bidder $-i$ drops out first and her drop-out price is less than $c-x$. For values y such that $\beta_{-i}^{-1}(c-x) < y < \beta_{-i}^{-1}(b)$, bidder $-i$ is still the first one to drop out, and her drop-out price determines whether bidder i will contribute the rest or not. For values y such that $y > \beta_{-i}^{-1}(b)$, there is provision as explained above. So, the payoff for bidder i is given by

$$\pi^i(x,b,\beta_{-i}) = \int_{\beta_{-i}^{-1}(c-x)}^{\beta_{-i}^{-1}(b)} [x-c+\beta_{-i}(y)]f(y)dy + (x-b)[1-F(\beta_{-i}^{-1}(b))].$$

Suppose instead of b, bidder i bids $b-\varepsilon$, where $\varepsilon > 0$. Then, the relevant payoff is given by

$$\pi^i(x,b-\varepsilon,\beta_{-i}) = \int_{\beta_{-i}^{-1}(c-x)}^{\beta_{-i}^{-1}(b-\varepsilon)} [x-c+\beta_{-i}(y)]f(y)dy + (x-b+\varepsilon)[1-F(\beta_{-i}^{-1}(b-\varepsilon))].$$

Then,

$$\begin{aligned}\pi^i(x,b-\varepsilon,\beta_{-i}) - \pi^i(x,b,\beta_{-i}) &= (x-b+\varepsilon)[1-F(\beta_{-i}^{-1}(b-\varepsilon))] - (x-b)[1-F(\beta_{-i}^{-1}(b))] \\ &\quad + \int_{\beta_{-i}^{-1}(c-x)}^{\beta_{-i}^{-1}(b-\varepsilon)} [x-c+\beta_{-i}(y)]f(y)dy - \int_{\beta_{-i}^{-1}(c-x)}^{\beta_{-i}^{-1}(b)} [x-c+\beta_{-i}(y)]f(y)dy \\ &\geq (x-b)[F(\beta_{-i}^{-1}(b)) - F(\beta_{-i}^{-1}(b-\varepsilon))] \\ &\quad +\varepsilon[1-F(\beta_{-i}^{-1}(b-\varepsilon))] - \int_{\beta_{-i}^{-1}(b-\varepsilon)}^{\beta_{-i}^{-1}(b)} [x-c+\beta_{-i}(\beta_{-i}^{-1}(b))]f(y)dy \\ &> 0.\end{aligned}$$

The weak inequality follows from the fact that $y < \beta_{-i}^{-1}(b)$. By Lemma 1, the strict inequality follows from the fact that $b \leq c/2$ and that $\varepsilon > 0$. Thus, the expected profit is strictly decreasing in b over the range $[b^{\beta_{-i}}, \overline{\beta}_i)$. Thus, whenever bidder i has a value $x < y^{\beta_{-i}}$ and $\beta_{-i}(.)$ is strictly increasing, she never makes a bid higher than $b^{\beta_{-i}}$. Now consider the case where $\beta_{-i}(y) = b$ for $y \in (y_1, y_2) \subseteq [y^{\beta_{-i}}, 1]$, for some y_1, y_2.[17] Again we consider two possible cases separately.

(i) $c-x > b \geq b^{\beta_{-i}}$: Over the range (y_1, y_2) both bidders have the same bid, so with probability 1/2 bidder i gets to be the first one to drop out and gets a payoff of $\int_{y_1}^{y_2} [x-b]f(y)dy$, and with probability 1/2 she gets to be the last one to drop out and gets a payoff of zero

[17]Since $b > b^{\beta_{-i}}$, we do not need to consider the case where bids coincide for values $y < y^{\beta_{-i}}$.

since $b < c - x$. Thus, bidder i's payoff when she bids b and her valuation is x, is

$$\pi^i(x, b, \beta_{-i}) = \frac{1}{2}\int_{y_1}^{y_2} [x - b] f(y) dy + \int_{y_2}^{1} [x - b] f(y) dy.$$

But, if she bids $b - \varepsilon$, for an arbitrarily small $\varepsilon > 0$, then her payoff is

$$\pi^i(x, b - \varepsilon, \beta_{-i}) = \int_{\widetilde{y}_1}^{y_2} [x - b + \varepsilon] f(y) dy + \int_{y_2}^{1} [x - b + \varepsilon] f(y) dy,$$

where $\widetilde{y}_1$ is such that $\beta^{-i}(\widetilde{y}_1) = b - \varepsilon$ if $\beta_{-i}(.)$ is continuous at y_1, and $\widetilde{y}_1 = y_1$ if $\beta_{-i}(.)$ is not continuous at y_1. For $\varepsilon > 0$ small enough, $\pi^i(x, b - \varepsilon, \beta_{-i})$ is strictly bigger than $\pi^i(x, b, \beta_{-i})$. Since any b such that $b > b^{\beta_{-i}}$ is strictly dominated by $b - \varepsilon$, any bid $b \in (c - x, b^{\beta_{-i}})$ cannot be optimal. Thus, bidder i bids $b^{\beta_{-i}}$.

(ii) $b \geq c - x$: Define y_0 such that $\beta_{-i}(y_0) = c - x$.[18] Then, bidder i's payoff when she bids b and her valuation is x, is

$$\begin{aligned}
\pi^i(x, b, \beta_{-i}) &= \int_{y_0}^{y_1} [x - c + \beta_{-i}(y)] f(y) dy + \frac{1}{2}\int_{y_1}^{y_2} [x - b] f(y) dy + \frac{1}{2}\int_{y_1}^{y_2} [x - c + b] f(y) dy \\
&\quad + \int_{y_2}^{1} [x - b] f(y) dy \\
&= \int_{y_0}^{y_1} [x - c + \beta_{-i}(y)] f(y) dy + \int_{y_1}^{y_2} [x - c/2] f(y) dy + \int_{y_2}^{1} [x - b] f(y) dy.
\end{aligned}$$

But, the bidder with value x can bid $b - \varepsilon$ and improve her payoff. To see this, let's first write the payoff from bidding $b - \varepsilon$:

$$\pi^i(x, b - \varepsilon, \beta_{-i}) = \int_{y_0}^{\widetilde{y}_1} [x - c + \beta_{-i}(y)] f(y) dy + \int_{\widetilde{y}_1}^{y_2} [x - b + \varepsilon] f(y) dy + \int_{y_2}^{1} [x - b + \varepsilon] f(y) dy,$$

where $\widetilde{y}_1$ is defined as in Case (i) above. If $\widetilde{y}_1 = y_1$, then $\pi^i(x, b - \varepsilon, \beta_{-i}) > \pi^i(x, b, \beta_{-i})$ since

$$\int_{\widetilde{y}_1}^{y_2} [x - b + \varepsilon] f(y) dy > \int_{y_1}^{y_2} [x - b] f(y) dy \geq \int_{y_1}^{y_2} [x - c/2] f(y) dy,$$

where the last inequality follows from $b \leq c/2$. If $\widetilde{y}_1 < y_1$, then $\pi^i(x, b - \varepsilon, \beta_{-i}) > \pi^i(x, b, \beta_{-i})$

[18] If there is more than one such y_0, then pick the smallest one. Also note that, the existence of y_0 is not guaranteed since $\beta_{-i}(.)$ is allowed to be discontinuous. Even so, we can define y_0 to be the value at which $\beta_{-i}(.)$ has a jump with the property that $\beta_{-i}(y) < c - x$ for all $y < y_0$, and $\beta_{-i}(y) > c - x$ for all $y > y_0$.

since

$$\begin{aligned}
&\int_{y_0}^{\widetilde{y}_1}[x-c+\beta_{-i}(y)]f(y)dy+\int_{\widetilde{y}_1}^{y_2}[x-b+\varepsilon]f(y)dy\\
>&\int_{y_0}^{\widetilde{y}_1}[x-c+\beta_{-i}(y)]f(y)dy+\int_{\widetilde{y}_1}^{y_2}[x-c/2]f(y)dy\\
>&\int_{y_0}^{y_1}[x-c+\beta_{-i}(y)]f(y)dy+\int_{y_1}^{y_2}[x-c/2]f(y)dy,
\end{aligned}$$

where the first inequality follows from the fact that $x-b+\varepsilon > x-c/2$, and the second inequality follows from the fact that $x-c/2 \geq x-c+\beta_{-i}(y)$ and $\widetilde{y}_1 < y_1$. Thus, any bid $b \geq c-x$ is strictly dominated by $b-\varepsilon$. Specifically $c-x$ is dominated by $c-x-\varepsilon$. Thus, we are back in Case (i), and the optimal bid is $b^{\beta_{-i}}$. Now consider the values $x > y^{\beta_{-i}}$. But in this case, there is only one region to check, which is $b > b^{\beta_{-i}}$, since $c-x < b^{\beta_{-i}}$. By the same argument in Case (ii) above, we can conclude that $\pi^i(x, b-\varepsilon, \beta_{-i}) > \pi^i(x, b, \beta_{-i})$, and hence any $b > b^{\beta_{-i}}$ is dominated by $b^{\beta_{-i}}$. ■

Proof of Lemma 5. First note that it is straightforward from Lemma 4 that $b^i = b^{\beta_{-i}}$. Thus, we have $\beta_i(x) \leq b^{\beta_{-i}}$ for all $x \in [0,1]$. Since $\beta_i(.)$ is non-decreasing, definition of $b^{\beta_{-i}}$ implies that $\beta_i(x) = b^{\beta_{-i}}$ for any $x > c - b^{\beta_{-i}} = y^{\beta_{-i}}$. So, we have shown that $\beta_i(x) = \beta_{-i}(y) = b$ for $x, y \in [c-b, 1]$, in an equilibrium with positive probability of provision with non-decreasing bidding functions (In fact, we have not shown that $\beta_i(c-b) = \beta_{-i}(c-b) = b$. But, if $\beta_i(c-b) < b$ and $\beta_{-i}(c-b) < b$, then one of the bidders deviates to $b-\varepsilon$, which will be clear below). We argue that this cannot be the case in an equilibrium with positive probability of provision , simply because bidder i would deviate to $b-\varepsilon$ for some x where $\varepsilon > 0$ is arbitrarily small. To see this, first note that bidder i with value x, when she bids b, gets a payoff of $x-b$ with probability 1/2 and a payoff of $x-c+b$ with probability 1/2. Note also that $x-b \geq x-c+b$ since $b \leq c/2$. But, dropping out at $b-\varepsilon$ generates a payoff of $x-b+\varepsilon$ over the range $[c-b+\varepsilon, 1]$, and zero over $[c-b, c-b+\varepsilon)$. So, over the range $[c-b, 1]$, the net gain from deviating from b to $b-\varepsilon$ is equal to

$$\int_{c-b+\varepsilon}^{1}[x-b+\varepsilon]f(y)dy - \{1/2\int_{c-b}^{1}[x-b]f(y)dy + 1/2\int_{c-b}^{1}[x-c+b]f(y)dy\}$$

$$= [x-b][1-F(c-b+\varepsilon)] - [x-c/2][1-F(c-b)] + \varepsilon[1-F(c-b+\varepsilon)].$$

This expression is positive for sufficiently small $\varepsilon > 0$ since $x-b \geq x-c/2$.[19] There is,

[19] If $b = c/2$, the net gain over the range $[c/2, 1]$ is equal to

$$\begin{aligned}
&[x-c/2][1-F(c/2+\varepsilon)] - [x-c/2][1-F(c/2)] + \varepsilon[1-F(c/2+\varepsilon)]\\
=\;&[F(c/2)-F(c/2+\varepsilon)][x-c/2] + \varepsilon[1-F(c/2+\varepsilon)],
\end{aligned}$$

however, a potential loss over the range $(\widetilde{y}, c-b)$ where $\widetilde{y}$ is defined to be the solution to $\beta_{-i}(y) = b - \varepsilon$ if there is a solution, and if there is no solution, $\widetilde{y}$ is such that $\beta_{-i}(y) < b - \varepsilon$ for all $y < \widetilde{y}$, and $\beta_{-i}(y) > b - \varepsilon$ for all $c - b > y > \widetilde{y}$. This loss, however, depends on $\beta_{-i}(y)$ for $y < c - b$, and there are three possible cases: (1) $\beta_{-i}(.)$ is not continuous at $c - b$. Then, $\beta_{-i}(y) < b$ for all $y < c - b$, which, together with discontinuity, implies $\widetilde{y} = c - b$. Thus, the loss is zero. Thus, the deviation to $b - \varepsilon$ is profitable for any $x \in [c-b, 1]$. (2) $\beta_{-i}(.)$ is continuous at $c - b$, and $\beta_{-i}(y) < b$ for all $y < c - b$. Then, the loss from deviation is equal to $\int_{\widetilde{y}}^{c-b} [x - c + \beta_{-i}(y)] f(y) dy$. But, for $x = c - b$, this expression is negative since $\beta_{-i}(y) < b$ for $y < c - b$. Thus, the deviation to $b - \varepsilon$ is profitable for $x = c - b$. (3) $\beta_{-i}(.)$ is continuous at $c - b$, and $\beta_{-i}(y) = b$ for $y \in (\widehat{y}, c - b)$, for some $\widehat{y} \in [0, c - b)$. Then, the loss is equal to $1/2 \int_{\widehat{y}}^{c-b} [x - c + b] f(y) dy$, which is zero when $x = c - b$. Thus, the deviation to $b - \varepsilon$ is profitable for $x = c - b$. ■

Proof of Lemma 6. Suppose that for bidder i we have, $x_1 < x_2$ and $\beta_i(x_1) > \beta_i(x_2)$, and bidder $-i$ bids according to $\beta_{-i}(y)$. We show that the bidding function $\beta_i(x)$ is not incentive compatible. More precisely, we show that $\pi^i(x_2, \beta_i(x_2), \beta_{-i}) \geq \pi^i(x_2, \beta_i(x_1), \beta_{-i})$ implies $\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i})$, and hence we will conclude that such $\beta_i(x)$ cannot be decreasing over some set of values. To see this, first define

$$
\begin{aligned}
Y_H &= \{y | \beta_{-i}(y) \geq \beta_i(x_1)\}, \\
Y_M &= \{y | \beta_i(x_1) > \beta_{-i}(y) \geq \beta_i(x_2)\}, \\
Y_L &= \{y | \beta_i(x_2) > \beta_{-i}(y)\}.
\end{aligned}
$$

There are five cases defined in terms of the relative magnitudes between $\beta_i(x_k)$ and $c - x_k$, where $k = 1, 2$.

(i) $c - x_1 > c - x_2 > \beta_i(x_2)$, $c - x_1 > \beta_i(x_1)$: The relevant payoffs are given as follows:

$$
\begin{aligned}
\pi^i(x_1, \beta_i(x_1), \beta_{-i}) &= \int_{Y_H \cap \{y | y \geq c - \beta_i(x_1)\}} [x_1 - \beta_i(x_1)] f(y) dy = [x_1 - \beta_i(x_1)] P_1, \\
\pi^i(x_2, \beta_i(x_2), \beta_{-i}) &= \int_{(Y_H \cup Y_M) \cap \{y | y \geq c - \beta_i(x_2)\}} [x_2 - \beta_i(x_2)] f(y) dy = [x_2 - \beta_i(x_2)] P_2, \\
\pi^i(x_1, \beta_i(x_2), \beta_{-i}) &= \int_{(Y_H \cup Y_M) \cap \{y | y \geq c - \beta_i(x_2)\}} [x_1 - \beta_i(x_2)] f(y) dy = [x_1 - \beta_i(x_2)] P_2, \\
\pi^i(x_2, \beta_i(x_1), \beta_{-i}) &= \int_{Y_H \cap \{y | y \geq c - \beta_i(x_1)\}} [x_2 - \beta_i(x_1)] f(y) dy + \int_{Y_M \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)] f(y) dy \\
&= [x_2 - \beta_i(x_1)] P_1 + \int_{Y_M \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)] f(y) dy,
\end{aligned}
$$

which is positive for $x = c/2$ and for sufficiently small $\varepsilon > 0$.

where $P_1 = \int_{Y_H \cap \{y | y \geq c - \beta_i(x_1)\}} f(y)dy$ and $P_2 = \int_{(Y_H \cup Y_M) \cap \{y | y \geq c - \beta_i(x_2)\}} f(y)dy$. Now, $\pi^i(x_2, \beta_i(x_2), \beta_{-i}) \geq \pi^i(x_2, \beta_i(x_1), \beta_{-i})$ implies

$$[x_2 - \beta_i(x_2)]P_2 \geq [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy,$$

which implies $[x_2 - \beta_i(x_2)]P_2 \geq [x_2 - \beta_i(x_1)]P_1$, or $P_2 \geq \frac{[x_2 - \beta_i(x_1)]}{[x_2 - \beta_i(x_2)]}P_1$.[20] Since $x_1 < x_2$, we have $P_2 > \frac{[x_1 - \beta_i(x_1)]}{[x_1 - \beta_i(x_2)]}P_1$. This follows from the fact that

$$\frac{d}{dx}\left(\frac{[x - \beta_i(x_1)]}{[x - \beta_i(x_2)]}\right) = \frac{[x - \beta_i(x_2) - x + \beta_i(x_1)]}{[x - \beta_i(x_2)]} = \frac{[\beta_i(x_1) - \beta_i(x_2)]}{[x - \beta_i(x_2)]^2} > 0.$$

$P_2 > \frac{[x_1 - \beta_i(x_1)]}{[x_1 - \beta_i(x_2)]}P_1$ implies $\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i})$.

Note that, we implicitly assumed that $P_1 > 0$. If $P_1 = 0$, then $\pi^i(x_1, \beta_i(x_1), \beta_{-i}) = 0$. Thus, the probability of provision is zero at x_1. Now, at this equilibrium with positive probability of provision, (β_i, β_{-i}), where the probability of provision is strictly positive, define x_i^* to be the infimum over the set of values of bidder i such that the probability of provision in this equilibrium is positive. Then, the probability of provision is strictly positive for all $x \in (x_i^*, 1]$ and positive only for those values. Thus, $x_1 < x^*$. Also, β_i must be non-decreasing in $(x^*, 1]$, otherwise we could apply the same argument in the interval $(x^*, 1]$ for those values exhibiting decreasing bidding behaviour, and get a higher x^* contradicting the definition of x^*. Since the probability of provision is zero for all $x \in [0, x_i^*]$, proofs of Lemma 4 and Lemma 5 can be used to show that this cannot be an equilibrium with positive probability of provision.

(ii) $c - x_1 > \beta_i(x_1) > \beta_i(x_2) > c - x_2$: The relevant payoffs are

$$\begin{aligned}
\pi^i(x_1, \beta_i(x_1), \beta_{-i}) &= [x_1 - \beta_i(x_1)]P_1, \\
\pi^i(x_2, \beta_i(x_2), \beta_{-i}) &= [x_2 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy, \\
\pi^i(x_1, \beta_i(x_2), \beta_{-i}) &= [x_1 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)]f(y)dy, \\
\pi^i(x_2, \beta_i(x_1), \beta_{-i}) &= [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cup (Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\})} [x_2 - c + \beta_{-i}(y)]f(y)dy.
\end{aligned}$$

The inequality $\pi^i(x_2, \beta_i(x_2), \beta_{-i}) \geq \pi^i(x_2, \beta_i(x_1), \beta_{-i})$ implies

$$[x_2 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy$$

[20] Note that $x_2 > \beta_i(x_2)$, simply because $x_2 > x_1 \geq \beta_i(x_1) > \beta_i(x_2)$.

$$\geq [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cup (Y_L \cap \{y|\beta_{-i}(y) \geq c - x_2\})} [x_2 - c + \beta_{-i}(y)]f(y)dy,$$

which implies $[x_2 - \beta_i(x_2)]P_2 \geq [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M} [x_2 - c + \beta_{-i}(y)]f(y)dy$. Thus, $[x_2 - \beta_i(x_2)]P_2 \geq [x_2 - \beta_i(x_1)]P_1$. By a similar argument in Case (i) above, we get $[x_1 - \beta_i(x_2)]P_2 > [x_1 - \beta_i(x_1)]P_1$, which in turn implies

$$[x_1 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y|\beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)]f(y)dy > [x_1 - \beta_i(x_1)]P_1,$$

that is, $\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i})$.[21]

(iii) $\beta_i(x_1) > c - x_1 > c - x_2 > \beta_i(x_2)$: The relevant payoffs are

$$\begin{aligned}
\pi^i(x_1, \beta_i(x_1), \beta_{-i}) &= [x_1 - \beta_i(x_1)]P_1 + \int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)]f(y)dy, \\
\pi^i(x_2, \beta_i(x_2), \beta_{-i}) &= [x_2 - \beta_i(x_2)]P_2, \\
\pi^i(x_1, \beta_i(x_2), \beta_{-i}) &= [x_1 - \beta_i(x_2)]P_2, \\
\pi^i(x_2, \beta_i(x_1), \beta_{-i}) &= [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy.
\end{aligned}$$

$\pi^i(x_2, \beta_i(x_2), \beta_{-i}) \geq \pi^i(x_2, \beta_i(x_1), \beta_{-i})$ implies

$$[x_2 - \beta_i(x_2)]P_2 \geq [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy,$$

which in turn implies

$$P_2 \geq \frac{[x_2 - \beta_i(x_1)]}{[x_2 - \beta_i(x_2)]} P_1 + \frac{1}{[x_2 - \beta_i(x_2)]} \int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy.$$

Since $x_2 > x_1$, the above inequality implies

$$P_2 > \frac{[x_1 - \beta_i(x_1)]}{[x_1 - \beta_i(x_2)]} P_1 + \frac{1}{[x_2 - \beta_i(x_2)]} \int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy.$$

Arranging terms and adding and subtracting $\int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)]f(y)dy$, we get,

$$\begin{aligned}
\pi^i(x_1, \beta_i(x_2)) > \pi^i(x_1, \beta_i(x_1)) + \tfrac{[x_1 - \beta_i(x_2)]}{[x_2 - \beta_i(x_2)]} \textstyle\int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy \\
- \textstyle\int_{Y_M \cap \{y|\beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)]f(y)dy.
\end{aligned}$$

[21] In case $P_1 = 0$, the argument given in Case (i) still applies.

We are done if we can show

$$\int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_2\}}\frac{[x_2-c+\beta_{-i}(y)]}{[x_2-\beta_i(x_2)]}f(y)dy \geq \int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_1\}}\frac{[x_1-c+\beta_{-i}(y)]}{[x_1-\beta_i(x_2)]}f(y)dy.$$

First, we show that for any set of values $Y\subseteq Y_M$, we have

$$\int_Y\frac{[x_2-c+\beta_{-i}(y)]}{[x_2-\beta_i(x_2)]}f(y)dy > \int_Y\frac{[x_1-c+\beta_{-i}(y)]}{[x_1-\beta_i(x_2)]}f(y)dy.$$

To see this,

$$\begin{aligned}\frac{d}{dx}\int_Y\frac{[x-c+\beta_{-i}(y)]}{[x-\beta_i(x_2)]}f(y)dy &= \int_Y\frac{d}{dx}\frac{[x-c+\beta_{-i}(y)]}{[x-\beta_i(x_2)]}f(y)dy\\ &= \int_Y\frac{[c-\beta_i(x_2)-\beta_{-i}(y)]}{[x-\beta_i(x_2)]^2}f(y)dy\\ &> 0.\end{aligned}$$

To see the last inequality, note that for any $y\in Y_M$ we have $\beta_{-i}(y)<\beta_i(x_1)\leq x_1$ and $\beta_i(x_2)\leq c-x_2<c-x_1$. Thus, for any $y\in Y_M$, we have $\beta_i(x_2)+\beta_{-i}(y)<c$. To conclude,

$$\begin{aligned}\int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_2\}}\frac{[x_2-c+\beta_{-i}(y)]}{[x_2-\beta_i(x_2)]}f(y)dy &> \int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_2\}}\frac{[x_1-c+\beta_{-i}(y)]}{[x_1-\beta_i(x_2)]}f(y)dy\\ &\geq \int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_1\}}\frac{[x_1-c+\beta_{-i}(y)]}{[x_1-\beta_i(x_2)]}f(y)dy.\end{aligned}$$

The last inequality above follows from $Y_M\cap\{y|\beta_{-i}(y)\geq c-x_1\}\subseteq Y_M\cap\{y|\beta_{-i}(y)\geq c-x_2\}$.[22]

(iv) $\beta_i(x_1)>c-x_1>\beta_i(x_2)>c-x_2$: The relevant payoffs are

$$\begin{aligned}\pi^i(x_1,\beta_i(x_1),\beta_{-i}) &= [x_1-\beta_i(x_1)]P_1+\int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_1\}}[x_1-c+\beta_{-i}(y)]f(y)dy,\\ \pi^i(x_2,\beta_i(x_2),\beta_{-i}) &= [x_2-\beta_i(x_2)]P_2+\int_{Y_L\cap\{y|\beta_{-i}(y)\geq c-x_2\}}[x_2-c+\beta_{-i}(y)]f(y)dy,\end{aligned}$$

$$\begin{aligned}\pi^i(x_1,\beta_i(x_2),\beta_{-i}) &= [x_1-\beta_i(x_2)]P_2,\\ \pi^i(x_2,\beta_i(x_1),\beta_{-i}) &= [x_2-\beta_i(x_1)]P_1+\int_{Y_M\cup(Y_L\cap\{y|\beta_{-i}(y)\geq c-x_2\})}[x_2-c+\beta_{-i}(y)]f(y)dy.\end{aligned}$$

[22] If $P_1=0$, then the proof still works because of the following strict inequality: $\int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_2\}}\frac{[x_2-c+\beta_{-i}(y)]}{[x_2-\beta_i(x_2)]}f(y)dy > \int_{Y_M\cap\{y|\beta_{-i}(y)\geq c-x_1\}}\frac{[x_1-c+\beta_{-i}(y)]}{[x_1-\beta_i(x_2)]}f(y)dy.$

The inequality $\pi^i(x_2, \beta_i(x_2), \beta_{-i}) \geq \pi^i(x_2, \beta_i(x_1), \beta_{-i})$ implies

$$\begin{aligned} & [x_2 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)] f(y) dy \\ \geq \quad & [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cup (Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\})} [x_2 - c + \beta_{-i}(y)] f(y) dy. \end{aligned}$$

That is,

$$P_2 \geq \frac{[x_2 - \beta_i(x_1)]}{[x_2 - \beta_i(x_2)]} P_1 + \frac{1}{[x_2 - \beta_i(x_2)]} \int_{Y_M} [x_2 - c + \beta_{-i}(y)] f(y) dy.$$

Since $x_2 > x_1$, we get

$$P_2 > \frac{[x_1 - \beta_i(x_1)]}{[x_1 - \beta_i(x_2)]} P_1 + \frac{1}{[x_2 - \beta_i(x_2)]} \int_{Y_M} [x_2 - c + \beta_{-i}(y)] f(y) dy.$$

After arranging terms, we add and subtract $\int_{Y_M \cap \{y | \beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)] f(y) dy$ to get

$$\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i}) + \frac{[x_1 - \beta_i(x_2)]}{[x_2 - \beta_i(x_2)]} \int_{Y_M} [x_2 - c + \beta_{-i}(y)] f(y) dy - \int_{Y_M} [x_1 - c + \beta_{-i}(y)] f(y) dy.$$

Since $\beta_i(x_2) < c - x_1$, for any $y \in Y_M$, we have $\beta_i(x_2) + \beta_{-i}(y) < c$. So, for any $Y \subseteq Y_M$,

$$\int_Y \frac{[x_2 - c + \beta_{-i}(y)]}{[x_2 - \beta_i(x_2)]} f(y) dy > \int_Y \frac{[x_1 - c + \beta_{-i}(y)]}{[x_1 - \beta_i(x_2)]} f(y) dy.$$

Thus, $\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i})$.[23]

(v) $\beta_i(x_1) > \beta_i(x_2) > c - x_1 > c - x_2$: Since $\overline{\beta}_i \leq c/2$ and $\beta_{-i}(y) < \beta_i(x_1)$ for any $y \in Y_M$, we have $\beta_i(x_2) + \beta_{-i}(y) < c$ for any $y \in Y_M$. Then, the argument given above in Case (iv) proves that $\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i})$. To see this,

$$\pi^i(x_1, \beta_i(x_1), \beta_{-i}) = [x_1 - \beta_i(x_1)]P_1 + \int_{Y_M \cup (Y_L \cap \{y | \beta_{-i}(y) \geq c - x_1\})} [x_1 - c + \beta_{-i}(y)] f(y) dy,$$

$$\pi^i(x_2, \beta_i(x_2), \beta_{-i}) = [x_2 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)] f(y) dy,$$

$$\pi^i(x_1, \beta_i(x_2), \beta_{-i}) = [x_1 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)] f(y) dy,$$

$$\pi^i(x_2, \beta_i(x_1), \beta_{-i}) = [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cup (Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\})} [x_2 - c + \beta_{-i}(y)] f(y) dy,$$

[23] If $P_1 = 0$, the argument given in the previous footnote works here too.

where $P_1 = \int_{Y_H \cap \{y | y \geq c - \beta_i(x_1)\}} f(y)dy$ and $P_2 = \int_{Y_M \cap \{y | y \geq c - \beta_i(x_2)\}} f(y)dy$. The inequality $\pi^i(x_2, \beta_i(x_2), \beta_{-i}) \geq \pi^i(x_2, \beta_i(x_1), \beta_{-i})$ implies

$$\begin{aligned} & [x_2 - \beta_i(x_2)]P_2 + \int_{Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\}} [x_2 - c + \beta_{-i}(y)]f(y)dy \\ \geq \quad & [x_2 - \beta_i(x_1)]P_1 + \int_{Y_M \cup (Y_L \cap \{y | \beta_{-i}(y) \geq c - x_2\})} [x_2 - c + \beta_{-i}(y)]f(y)dy. \end{aligned}$$

That is,

$$P_2 \geq \frac{[x_2 - \beta_i(x_1)]}{[x_2 - \beta_i(x_2)]} P_1 + \frac{1}{[x_2 - \beta_i(x_2)]} \int_{Y_M} [x_2 - c + \beta_{-i}(y)]f(y)dy.$$

Since $x_2 > x_1$, we get

$$P_2 > \frac{[x_1 - \beta_i(x_1)]}{[x_1 - \beta_i(x_2)]} P_1 + \frac{1}{[x_2 - \beta_i(x_2)]} \int_{Y_M} [x_2 - c + \beta_{-i}(y)]f(y)dy.$$

After arranging terms, we add and subtract $\int_{Y_M \cap \{y | \beta_{-i}(y) \geq c - x_1\}} [x_1 - c + \beta_{-i}(y)]f(y)dy$ to get

$$\pi(x_1, \beta(x_2), \beta_{-i}) \geq [x_1 - \beta(x_2)]P_2$$

$$> \pi(x_1, \beta(x_1), \beta_{-i}) + \frac{[x_1 - \beta(x_2)]}{[x_2 - \beta(x_2)]} \int_{Y_M} [x_2 - c + \beta_{-i}(y)]f(y)dy - \int_{Y_M} [x_1 - c + \beta_{-i}(y)]f(y)dy.$$

Since $\overline{\beta}_i \leq c/2$ and $\beta_{-i}(y) < \beta_i(x_1)$ for any $y \in Y_M$, we have $\beta_i(x_2) + \beta_{-i}(y) < c$ for any $y \in Y_M$. Thus, for any $Y \subseteq Y_M$, we have

$$\int_Y \frac{[x_2 - c + \beta_{-i}(y)]}{[x_2 - \beta_i(x_2)]} f(y)dy > \int_Y \frac{[x_1 - c + \beta_{-i}(y)]}{[x_1 - \beta_i(x_2)]} f(y)dy.$$

Thus, as in Case (iv) we have $\pi^i(x_1, \beta_i(x_2), \beta_{-i}) > \pi^i(x_1, \beta_i(x_1), \beta_{-i})$. ■

Proof of Proposition 1. Pick a non-decreasing and symmetric equilibrium of the endogenous order mechanism. Pick an arbitrary pair of valuations x and y such that $x + y > c$.[24] Denote the contribution function of the first contributor in the random order mechanism with $\tau(\cdot)$, and the contribution function of the winner of the auction in the endogenous order mechanism with $\kappa(\cdot)$, dropping b_1 and b_2 to save space.

Case 1: *There is no provision in the random order mechanism regardless of the order.* Then, clearly endogenous order mechanism cannot do worse, simply because the probability of provision cannot be negative.

Case 2: *There is provision in the random order mechanism regardless of the order.* Then

[24] If $x + y < c$, then the probability of provision is zero in both mechanisms. We ignore $x + y = c$ because it is a measure zero event.

there is also provision in the endogenous order mechanism. To see this, consider three subcases regarding possible information revelation in the endogenous order mechanism.

Case 2.1: The winner infers nothing about the valuation of the loser. Then, the winner, after paying b_1 (the bid of the loser) will contribute $\max\{\tau(x) - b_1, 0\}$ in the second stage; that is, his total contribution will be at least $\tau(x)$. So, if there is provision in the random order mechanism, there is also provision in the equilibrium of the endogenous order mechanism.

Case 2.2: The winner learns the exact value of the loser. Then the winner will contribute just as enough to make the loser pay the rest in the second stage. So, if there is provision in the random order mechanism, there is also provision in the equilibrium of the endogenous order mechanism.

Case 2.3: The winner infers that the valuation of the loser lies in a subinterval $[a, d]$ of $[0, 1]$ where $0 \leq a < d \leq 1$. That is, the equilibrium bidding function of the loser is constant over the interval $[a, d]$. Since we are looking at the equilibrium path, Bayes' rule implies that the posterior is

$$\widehat{f} = \begin{cases} zf(s) & \text{if } a \leq s \leq d \\ 0 & \text{otherwise} \end{cases}$$

where $z = 1/\int_a^d f(s)ds > 1$. Consider three subcases.

Case 2.3.1: Suppose that $c - \tau(x) \geq d$. Then, the probability of provision in the random order mechanism is zero. The probability of provision in the equilibrium of the endogenous order is positive since $c - \kappa(x) \leq d$.

Case 2.3.2: Suppose that $a < c - \tau(x) < d$. That is, $c - d < \tau(x) < c - a$. Note that $\tau(x)$ satisfies the first order condition

$$(x - \tau(x))f(c - \tau(x)) - \int_{c-\tau(x)}^1 f(s)ds = 0$$

Hence,

$$\begin{aligned} (x - \tau(x))\widehat{f}(c - \tau(x)) - \int_{c-\tau(x)}^d \widehat{f}(s)ds &= z\{(x - \tau(x))f(c - \tau(x)) - \int_{c-\tau(x)}^d f(s)ds\} \\ &= z\int_d^1 f(s)ds \\ &\geq 0 \end{aligned}$$

Then, the optimal $\kappa(x)$ is at least as big as $\tau(x)$. Thus if there is provision in the random order mechanism then there is also provision in the equilibrium of the endogenous order mechanism.

Case 2.3.3: Suppose that $c - \tau(x) \leq a$. That is, $c - d < c - a \leq \tau(x)$. Then for any

$\kappa \in [c-d, c-a]$, we have $\kappa \leq \tau(x)$, so the strict concavity of the ex ante objective function implies

$$(x-\kappa)f(c-\kappa) - \int_{c-\kappa}^{1} f(s)ds > 0$$

Hence,

$$\begin{aligned}
(x-\kappa)\widehat{f}(c-\kappa) - \int_{c-\kappa}^{d} \widehat{f}(s)ds &= z\{(x-\kappa)f(c-\kappa) - \int_{c-\kappa}^{d} f(s)ds\} \\
&> z\int_{d}^{1} f(s)ds \\
&\geq 0
\end{aligned}$$

Thus, the optimal $\kappa(x)$ is a corner solution, which is $\kappa(x) = c-a$. That is, $c-\kappa(x) = a$. Thus, there is provision in the endogenous order mechanism because $y \in [a, d]$ implies $y \geq c - \kappa(x)$.

Case 3: *There is provision in the random order mechanism for one order but not the other.* Without loss of generality, suppose there is provision when the bidder with valuation x contributes first but not when the bidder with valuation y contributes first. Thus the probability of provision in the random order mechanism given (x, y) is 1/2. We first prove that $x > y$. Let $\tau(z)$ be the contribution of the first player in the random order mechanism, characterized by

$$(z - \tau(z))f(c - \tau(z)) = 1 - F(c - \tau(z))$$

Define $G(z, \tau) = (z-\tau)f(c-\tau) - [1 - F(c-\tau)] = 0$. Then, by implicit function theorem

$$\frac{d\tau}{dz} = -\frac{\partial G(z,\tau)/\partial z}{\partial G(z,\tau)/\partial \tau} = -\frac{f(c-\tau)}{\partial G(z,\tau)/\partial \tau}$$

Note that $\partial G(z,\tau)/\partial\tau < 0$ because it is the second derivative of the ex ante objective function which is strictly concave by the assumption $\frac{2f}{|f'|} > 2 - c$. If $|\partial G(z,\tau)/\partial\tau| < f(c-\tau)$, then $\frac{d\tau}{dz} > 1$. That is, $|-f'(c-\tau)(z-\tau) - 2f(c-\tau)| < f(c-\tau)$ implies $\frac{d\tau}{dz} > 1$. Since $\partial G(z,\tau)/\partial\tau < 0$, we have $\frac{d\tau}{dz} > 1$ if $f'(c-\tau)(z-\tau) + f(c-\tau) < 0$; that is, $\frac{f(c-\tau)}{|f'(c-\tau)|} < z - \tau$ since $f' < 0$. Since $2 - c > \frac{f}{|f'|}$ implies $\frac{f(c-\tau)}{|f'(c-\tau)|} < z - \tau$, we get $\frac{d\tau}{dz} > 1$. Since there is no provision when bidder with valuation y moves first we have $x + \tau(y) < c$. Also since there is provision when bidder with valuation x moves first, we have $y + \tau(x) \geq c$. Thus, $y + \tau(x) > x + \tau(y)$. That is, $\tau(x) - x > \tau(y) - y$ which implies $x > y$ since $\tau(z) - z$ is strictly increasing.

Since $x > y$ and the equilibrium is non-decreasing and symmetric, $\beta(x) \geq \beta(y)$. If $\beta(x) > \beta(y)$, then bidder with valuation x wins and argument in case 2 applies, thus there is provision. If $\beta(x) = \beta(y)$, then the bidder with valuation x wins with probability 1/2. There is provision if she wins the auction; that is, the provision probability in the endogenous order mechanism given (x, y) is 1/2.

So we have shown that in all cases the probability of provision given (x, y) in the endogenous order mechanism is at least as big as the probability of provision in the random order mechanism, given this (x, y). ■

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