Economic Growth and Solow Model

“Is there some action a government of India could take that would lead the Indian economy to grow like Indonesia's or Egypt's? If so, what exactly? If not, what is it about the nature of India that makes it so? The consequences for human welfare involved in questions like these are simply staggering: Once one starts to think about them, it is hard to think about anything else.”

1 Economic Growth

One of the most striking features of the world economy is the vast disparity in standards of living and rates of economic growth. For example, in 2000, real GDP per capita in the United States was more than fifty times that in Ethiopia. And over the period 1975-2003, real GDP per capita in China grew at a rate of 7.6% annually, while, in Argentina, real GDP per capita grew at a rate of only 1.8%: Seventy-six times slower. Moreover, there are often vast reversals in prosperity over time. Argentina, Venezuela, Uruguay, Israel, and South Africa were in the top 25 countries (by GDP per capita) in 1960, but none made it to the top 25 in 2000. From 1960 to 2000, the fastest growing country in the world was Taiwan, which grew at 6%. The slowest growing country was Zambia which grew at -1.8%. That is, people in Zambia were markedly worse off in 2000 than they were in 1960. The theory of economic growth seeks to address these issues and provide explanations.

The Solow model is a long-run model that seeks to explain why there are such vast income disparities and growth differences across the world. It describes the evolution of potential output or the productive capacities of countries over time. Because we assume an economy is always at potential in the long run, there are no recessions or booms in this analysis. Furthermore, there is no mention of nominal quantities such as money and prices as the classical dichotomy holds; namely, money has no effect on output in the long run and is thus irrelevant for explaining output differences. Over the long run, printing pieces of paper cannot generate increases in prosperity.

In 1957, Nicholas Kaldor documented some key facts on economic growth by empirical investigation. Kaldor did not claim that any of the variables he examines would be constant at all times; on the contrary, growth rates and income shares fluctuate strongly over the business cycle. Instead, his claim was that these quantities tend to be constant when averaging the data over long periods of time. His broad generalizations, which were initially derived from U.S. and U.K. data, but were later found to be true for many other countries as well, came to be known as 'stylized facts'.
These may be summarized and related as follows:

1. Output per worker grows at a roughly constant rate that does not diminish over time.
2. Capital per worker grows over time.
3. The rate of return to capital is constant.
4. The capital/output ratio is roughly constant.
5. The share of capital and labor in net income are nearly constant.
6. Growth rate of output per worker differs substantially across countries.

Suppose \( Y \) denotes output, \( L \) denotes labor, \( K \) denotes capital, \( r \) denotes real rate of return, and \( w \) denotes real wage. Then Kaldor’s stylized facts can be described as follows:

1. \( Y/L \) grows over time at a constant rate.
2. \( K/L \) grows over time.
3. \( r \) roughly constant over time.
4. \( K/Y \) roughly constant over time.
5. \( rK/Y \) and \( wL/Y \) roughly constant over time.
6. Growth rates of \( Y/L \) differ substantially across countries.

In late 1950s, Robert Solow came with a model that captures most of these stylized facts. The Solow Model (also known as the Solow-Swan Model by Trevor Swan) became the workhorse model for analyzing economic growth for a long period of time. The main reason why the model has been used very extensively is that the model is very simple and tractable, yet powerful enough to explain many long-run key facts.

2 The Solow Model

The model assumes that a country’s output is produced by a representative neoclassical production function.

\[
Y_t = F(A, K_t, L_t)
\]

where \( Y \) denotes output, \( L \) denotes labor, \( K \) denotes capital, and \( A \) denotes a measure of productivity (also known as the Solow-residual). A Cobb-Douglas production function can be a good example of a neoclassical production function due to the fact that it satisfies the below required properties (A Cobb-Douglas production function is \( Y_t = F(K_t, L_t) = AK_t^\alpha L_t^{1-\alpha} \)). \( A \) is a given (exogenous) technology that cannot controlled or changed by a country, and for simplicity it is assumed to be constant, and we will omit it using as an argument when representing the production function, although formally it still is an argument).

The neoclassical production function to has to have all of the following properties:

1. Constant returns to scale (CRS) in capital \( K \), and labor \( L \):
   
   \[
   F(\lambda K, \lambda L) = \lambda F(K, L) \quad \text{for all } \lambda > 0.
   \]
2. Positive and diminishing marginal products of capital and labor:
\[ F_K(K, L) > 0; \quad F_L(K, L) > 0, \quad F_{kk}(K, L) < 0; \quad F_{ll}(K, L) < 0. \]

3. Essentiality of inputs:
\[ F(0, L) = 0, \quad F(K, 0) = 0. \]

4. Inada (limiting) conditions:
\[ \lim_{K \to 0} F_K(K, L) = \infty \]
\[ \lim_{K \to \infty} F_K(K, L) = 0 \]
\[ \lim_{L \to 0} F_L(K, L) = \infty \]
\[ \lim_{L \to \infty} F_L(K, L) = 0 \]

The first property implies that if, for instance, we double the inputs capital and labor, the output doubles, as well. It can be interpreted as we can open another factory that uses the same technology, and produce at this factory at the same production possibility frontier.

Second property implies that if we increase either of the inputs, output increases, yet only at a decreasing rate: As we use more and more of an input, its contribution to output decreases: As we use a lot of capital and too little labor, additional units of capital increases output only slightly.

Third property implies that both capital and labor are essential in production: If we lack either of them, we cannot produce anything at all.

The fourth property, named after Japanese economist Ken-Ichi Inada, implies that if we have no capital or labor, the first unit of the inputs increase output substantially since the lack of the input implies zero production, and a slight increase in the input contributes to a strictly positive production. It also suggests that as we increase one of the inputs so much without increasing the other one, after a point (infinite units to be exact) the additional units of the vastly abundant input do not increase production at all.

Using these properties of the production function, we will be able to answer most of the questions we are interested in.

Before going on, it is important to note that we are not really interested in aggregate income/output, \( Y_t \), in this world. What we are really concerned with is income per capita, \( Y_t / L_t \), which is a conventional measure of standard of living. This shows how well each of us, on average, is doing.

We will assume that everybody works, and the number of workers is equal to the number of citizens of a country. Therefore \( L_t \) is both the number of workers, and the number of citizens, and this is why \( Y_t / L_t \) is indeed income/output per capita. Further for now, we will assume that population is constant, hence \( L_t = L_{t+1} = L_{t+2} = \ldots \).
Since we are interested in per capita output, and not aggregate output, we want to convert aggregate production \( Y_t = F(K_t, L_t) \) into per-capita production, and call \( y_t = f(K_t, L_t) \). First, let’s define per-capita output, simply as total output divided by the number of citizens \( y_t = \frac{Y_t}{L_t} \).

Since \( Y_t = F(K_t, L_t) \), then we can write \( y_t = \frac{F(K_t, L_t)}{L_t} \). Next, we will use the first property of the production function. We know that \( F(\lambda K, \lambda L) = \lambda F(K, L) \). We can pick \( \lambda = \frac{1}{L_t} \) since it holds for any \( \lambda \). First note that if we pick \( \lambda = \frac{1}{L_t} \), then we can write \( F(\lambda K_t, \lambda L_t) = F\left(\frac{K_t}{L_t}\right, 1\)\). This function depends not on the level of aggregate capital or labor, but per-capita capital \( \frac{K_t}{L_t} \).

Further, by using property 1, \( F(\lambda K, \lambda L) = \lambda F(K, L), \lambda = \frac{1}{L_t} \) implies

\[
F(\lambda K_t, \lambda L_t) = F\left(\frac{K_t}{L_t}\right, 1\) = \frac{1}{L_t} F(K_t, L_t) = \frac{Y_t}{L_t},
\]

which is the definition of \( y_t \).

Then, we can write \( y_t = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}\right, 1\) \). Let’s call per-capita capital \( \frac{K_t}{L_t} \) as \( k_t \), i.e. \( \frac{K_t}{L_t} = k_t \) and per-capita output \( y_t = \frac{F(K_t, L_t)}{L_t} = \frac{F(\frac{K_t}{L_t})}{L_t} = f\left(\frac{K_t}{L_t}\right) = f(k_t) \) so that we would have \( y_t = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}\right, 1\) = \( f\left(\frac{K_t}{L_t}\right) = f(k_t) \). We will work with \( f(k_t) \) instead of \( F(K_t, L_t) \) since our focus is on per-capita, but not aggregate units.

If our aggregate production function is a Cobb-Douglas production function, then \( Y_t = F(K_t, L_t) = A K_t^\alpha L_t^{1-\alpha} \). To convert aggregate production into per-capita production, we simply divide production by \( L \), so we can derive \( f(k_t) \) as follows:

\[
f(k_t) = \frac{F(K_t, L_t)}{L_t} = \frac{AK_t^\alpha L_t^{1-\alpha}}{L_t} = \frac{AK_t^\alpha}{L_t^\alpha} = \frac{AK_t^\alpha}{L_t^\alpha} = A \left(\frac{K_t}{L_t}\right)^\alpha = Ak_t^\alpha
\]

Hence \( y_t = f(k_t) = Ak_t^\alpha \) as desired, so that per-capita output depends on per-capita capital, but not aggregate capital. Note that \( 0 < \alpha < 1 \) is a general property of Cobb-Douglas production functions (e.g. for the U.S., the estimate value is around 0.3). Then, for a constant level of \( A=1 \), \( f(k_t) = Ak_t^\alpha = k_t^\alpha \) would look like follows:
As the value for the constant $A$ goes up, $f(k)$ takes higher values and lies above initial graph, and as $A$ goes down, $f(k)$ takes lower values and lies below initial graph.

The per-capita production function $f(k)$ preserves some of the properties of the aggregate production function $F(K, L)$. The important properties of $f(k)$ can be listed as follows:

1. $\frac{\partial Y}{\partial K} = \frac{\partial F(K, L)}{\partial K} = \frac{\partial f(k)}{\partial k} = f'(k)$

Proof: We will invoke chain rule and first property of $F(K, L)$ here. First, note that $F(K, L) = L F(\frac{K}{L})$ due to constant returns to scale technology property. Further, note that $F(\frac{K}{L}) = f(k)$. Then by using chain rule in calculus, we can write the above equation as
follows: \[ \frac{\partial F(K,L)}{\partial K} = \frac{\partial Lf(K,L)}{L} \frac{dk}{dK} = \frac{\partial Lf(k)}{dk} \frac{dk}{dK} = Lf'(k) \frac{dk}{dK}. \] Since \[ k = \frac{K}{L}, \quad \frac{dk}{dK} = \frac{1}{L} \].

Plugging this, we have \[ \frac{\partial F(K,L)}{\partial K} = Lf'(k) \frac{dk}{dK} = Lf'(k) \frac{1}{L} = f'(k) \] as stated.

ii. \( f'(k) > 0, \quad f''(k) < 0 \)

Proof: We already showed that \( f'(k) = F'_k(K,L) \). Since \( F'_k(K,L) > 0 \) by the second property, then \( f'(k) > 0 \) immediately follows as well. The second inequality \( f''(k) < 0 \) follows from the fact that \( f''(k) = F''_{kk}(K,L) \times L \), and \( F''_{kk}(K,L) < 0 \) & \( L > 0 \).

iii) \( f(0) = 0 \)

Proof: \( k = 0 \) requires \( k = \frac{K}{L} = 0 \), which then implies \( K = 0 \). We know, by property 3, that \( F(0,L) = 0 \). Then \( f(k) = \frac{F(0,L)}{L} = 0 \) holds.

iv) \( \lim_{k \to 0} f'(k) = \infty \) and \( \lim_{k \to \infty} f'(k) = 0 \)

Proof: This result immediately is immediately followed by i) and property 4.

The first property of \( f(k) \) tells us that the marginal product of capital on aggregate output is precisely equal to marginal product of per-capita capital on per-capita output.

The second property of \( f(k) \) tells us that as per-capita capital increases, so does per-capita output, yet only at a diminishing rate.

The third property of \( f(k) \) tells us that 0 units of per-capita capital produces 0 units of per-capita output.

The fourth property of \( f(k) \) tells us that marginal product of per-capita capital goes to infinity as per-capita capital goes to 0, and marginal product of per-capita capital goes to 0 as per-capita capital goes to 0, which is another (and more strict) restriction on diminishing marginal product of per-capita capital.

The model assumes that consumers consume a fixed fraction of total output, and save the rest. The idea is similar to marginal propensity to consume and marginal propensity to consume. If saving rate is constant at \( s \), then total savings in the economy would be \( S_t = sY_t \). Total Consumption then would be simply total output minus total savings i.e. \( C_t = Y_t - sY_t = Y_t(1-s) \).

What happens to savings? They are converted into investment in the form of \( \text{capital} \). That is \( I_t = S_t = sY_t \).

We will further assume that capital depreciates, i.e. gets useless at a constant rate of \( \delta \) each period. Hence, if we have \( K_t \) units of aggregate capital at period \( t \), next period only \((1-\delta)K_t\) of it will be carried over.
By using these two facts, we can write down the connection between $K_t$ and $K_{t+1}$, which we will call as the ‘law of motion’ for aggregate capital. The law of motion can be written as:

$$K_{t+1} = sY_t + (1 - \delta)K_t = sF(K_t, L_t) + (1 - \delta)K_t$$

The first element in the right-hand side is the savings converted to capital, and second element is the capital which survives depreciation.

As mentioned earlier, we do not like dealing with aggregate quantities, and our main focus is on per-capita quantities. So we want to convert the law of motion for aggregate capital to law of motion for per-capita capital.

How can we do that? First note that, to convert $K_{t+1}$ into per-capita capital, we need to divide it by $L_{t+1}$; and to convert the variables on the right hand side into per-capita terms, we need to divide the terms by $L_t$. First, let’s divide all terms in the law of motion by $L_{t+1}$:

$$\frac{K_{t+1}}{L_{t+1}} = \frac{sY_t + (1 - \delta)K_t}{L_{t+1}}.$$ Since $k_{t+1} = \frac{K_{t+1}}{L_{t+1}}$ by definition, we can write the equation as

$$k_{t+1} = \frac{sY_t + (1 - \delta)K_t}{L_{t+1}}.$$ Next we divide and multiply all terms in the right hand side by $L_t$

$$k_{t+1} = \frac{sY_t + (1 - \delta)K_t}{L_t} \cdot \frac{L_t}{L_{t+1}}.$$ Recall that we defined $y_t = \frac{Y_t}{L_t}$ and $k_t = \frac{K_t}{L_t}$. Then we can write the equation as:

$$k_{t+1} = \left[sy_t + (1 - \delta)k_t\right] \cdot \frac{L_t}{L_{t+1}}.$$ Finally, remember that labor/citizen population is constant $L_{t+1} = L_t$, therefore $\frac{L_t}{L_{t+1}} = 1$. Then, we can rewrite the equation as follows:

$$k_{t+1} = sy_t + (1 - \delta)k_t = sf(k_t) + (1 - \delta)k_t$$

This is the law of motion for per-capita capital. If $y_t = f(k_t) = Ak_t^{\alpha}$, then the law of motion would be $k_{t+1} = sA_k^{\alpha} + (1 - \delta)k_t$. So by observing the values for the parameters $s, \alpha, \delta$ and $n$, and the per-capita capital level $k_t$, we know precisely the level of per-capita capital for the next period, $k_{t+1}$. 
We know that the level of per-capita investment in the economy is a fraction of per-capita output, and we know that depreciating capital is only a constant $\delta$ fraction of the existing per-capita capital. Let's analyze these on a graphical illustration:

We have $k$ on the horizontal and $y$ on the vertical axis. Production at a specific capital level $k$ is simply $f(k)$ and saving is a constant fraction of it $sf(k)$. The level of investment is also $sf(k)$. Depreciation is a constant fraction of existing per-capita capital i.e. $\delta k$, which is lost from the economy. If the level of investment exceeds depreciated per-capita capital, then per-capita capital increases. If, on the other hand, depreciated per-capita capital exceeds per-capita the level of investment then per-capita capital decreases. If we focus on a point left to $k^*$, we see that investment exceeds depreciation, hence capital increases. If we focus on a point right to $k^*$, we see that depreciation exceeds investment, thus capital decreases.

A steady-state is a point where all per-capita variables are constant at all times. With that definition, we observe that $k^*$ is our steady-state level of per-capita capital. Why? Because at this point, and only this point, investment is precisely equal to depreciation, thus capital level
remains unchanged. As per-capita capital remains unchanged, so does per-capita output $y^* = f(k')$. Also, saving and consumption are just a constant fractions of per-capita output $y^*$, so they remain unchanged. Hence, this point where all per-capita variables remain unchanged is our steady-state. Again, note that at this point, $sf(k) = \delta k$.

Let’s analyze the model using a different graph. This time we will have $k_t$ on the horizontal and $k_{t+1}$ on the vertical axis, and we will make use of the 45°-line. Remember, we derived the law of motion for per-capita capital as follows:

$$k_{t+1} = sy_t + (1-\delta)k_t = sf(k_t) + (1-\delta)k_t$$

Let’s call $g(k_t)$ as $k_{t+1}$, i.e. $g(k_t) = k_{t+1} = sy_t + (1-\delta)k_t = sf(k_t) + (1-\delta)k_t$

If we start at a low capital point at period 0, $k_0$, next period’s capital level $g(k_0) = k_1$ will be higher than today’s capital $k_0$. Therefore, capital increases and goes up to $k_1$ next period. At period 1, now the capital level is $k_1$ and $g(k_1) = k_2$ is still greater than $k_1$, hence capital increases up to $k_2$ next period. As long as $g(k_1) = k_{t+1}$ is greater than $k_t$ (which are the points where $g(k_t) = k_{t+1}$ lies above the 45°-line) capital keeps increasing. The point where $g(k_t) = k_{t+1} = k_t$, capital remains unchanged, and we have our steady-state.

Note that this point is exactly the same $k^*$ we derived before, and at this point, $sf(k) = \delta k$.
Can we find the values for $k^*$ and $y^*$? Yes, we can, by using only the law of motion for per-capita capital and the definition of the steady-state. Remember

\[ k_{i+1} = sf(k_i) + (1-\delta)k_i \] is our law of motion. At steady-state, \( k_i = k_{i+1} = k_{i+2} = \ldots = k^* \).

Then, the steady-state is the point where the law of motion can be rewritten as \( k^* = sf(k') + (1-\delta)k^* \).

If our production function is Cobb-Douglas with constant technology, i.e. \( y_t = f(k_t) = Ak_t^\alpha \), then \( f(k^*) = Ak^\alpha \). Then our steady-state is given as the point where

\[ k^* = sAk^\alpha + (1-\delta)k^* \]

satisfied. Dividing all terms by \( k^* \) and rearranging,

\[ sAk^{\alpha-1} = \delta \].

If we leave \( k^* \) alone, we would have \( k^* = \left(\frac{\delta}{sA}\right)^{1-\alpha} \) as our steady-state level of capital. If we know this value, we can easily find per-capita output \( f(k^*) = Ak^\alpha = A\left(\frac{\delta}{sA}\right)^{\frac{\alpha}{\alpha-1}} \).

Steady-state per-capita saving is just a fraction of steady-state per-capita output, which is \( sA\left(\frac{\delta}{sA}\right)^{\frac{\alpha}{\alpha-1}} \) and steady-state consumption is simply per-capita output minus savings, i.e. \( (1-s)A\left(\frac{\delta}{sA}\right)^{\frac{\alpha}{\alpha-1}} \) or equivalently \( A\left(\frac{\delta}{sA}\right)^{\frac{\alpha}{\alpha-1}} - sA\left(\frac{\delta}{sA}\right)^{\frac{\alpha}{\alpha-1}} \).

Therefore, we are able to find out the exact level of steady-state variables if we are given a functional form, and the values for the parameters.
3. Comparative Statics

a) Depreciation rate $\delta$
What happens when the depreciation rate $\delta$ increases? Let $\delta_2 > \delta_1$.

As shown in the figure, $\delta k$ gets steeper, and the point where $\delta k$ intersects $sf(k)$ curve moves to the left, implying lower level of steady-state per-capita capital, and accordingly per-capita output $y^* = f(k^*)$.

Homework: If the production function is given as Cobb-Douglas, can we verify this result that higher depreciation rate decreases steady-state level of per-capita capital by using the steady-state analytical closed form solution $f(k^*) = Ak^\alpha = A \left( \frac{\delta}{sA} \right)^{\frac{\alpha}{\alpha-1}}$? If so, how?

b) Saving rate $s$
What happens saving rate $s$ increases? Let $s_2 > s_1$. 

As shown in the figure, $s_2 f(k)$ lies above $s_1 f(k)$ at all points, and the point where $\delta k$ intersects $sf(k)$ curve moves to the right, implying higher level of steady-state per-capita capital, and accordingly per-capita output $y^* = f(k^*)$.

Homework: If the production function is given as Cobb-Douglas, can we verify this result that higher saving rate increases steady-state level of per-capita capital by using the steady-state analytical closed form solution $f(k^*) = Ak^{\alpha} = A\left(\frac{\delta}{sA}\right)^{\frac{\alpha}{\alpha-1}}$? If so, how?

4. Solow Model with Population Growth

Assuming a constant population, i.e. zero population growth rate, is a very restrictive assumption, and ignores a potential reason of the discrepancy of economic growth across countries: Different population growth rates. To analyze if population growth rate matters, in this section we will relax the assumption that population remains constant. Now, we will assume that population grows at a constant rate $n$, which is given (exogenous), hence $L_{t+1} = (1+n)L_t$. All remaining features of the model will be identical to zero-population-growth case in section 2.

First, since the model is identical except for the growth rate of labor, we can write down the same law of motion for aggregate capital just as in section 2:

$$K_{t+1} = sY_t + (1-\delta)K_t = sF(K_t, L_t) + (1-\delta)K_t$$

As mentioned already, we do not like dealing with aggregate quantities, and our main focus is on per-capita quantities. So we want to convert the law of motion for aggregate capital to law of motion for per-capita capital.
How can we do that? First note that, to convert $K_{t+1}$ into per-capita capital, we need to divide it by $L_{t+1}$, and to convert the variables on the right hand side into per-capita terms, we need to divide the terms by $L_t$. First, let’s divide all terms in the law of motion by $L_{t+1}$:

$$\frac{K_{t+1}}{L_{t+1}} = \frac{sy_i + (1-\delta)K_i}{L_{t+1}}.$$ Since $k_{t+1} = \frac{K_{t+1}}{L_{t+1}}$, by definition, we can write the equation as

$$k_{t+1} = \frac{sy_i + (1-\delta)K_i}{L_{t+1}}.$$ Next we divide and multiply all terms in the right hand side by $L_t$:

$$k_{t+1} = \frac{sY_i + (1-\delta)K_i}{L_t}.$$ Recall that we defined $y_i = \frac{Y_i}{L_t}$ and $k_i = \frac{K_i}{L_t}$. Then we can write the equation as:

$$k_{t+1} = [sy_i + (1-\delta)k_i] \frac{L_t}{L_{t+1}}.$$ Finally, remember that labor/citizen grows at a constant rate

$$L_{t+1} = (1+n)L_t,$$ therefore

$$\frac{L_t}{L_{t+1}} = \frac{1}{1+n}.$$ Then, we can rewrite the equation as follows:

$$k_{t+1} = \left[\frac{sy_i + (1-\delta)k_i}{1+n}\right].$$

This is the law of motion for per-capita capital. If $y_i = f(k_i) = Ak_i^\alpha$, then the law of motion would be

$$k_{t+1} = \left[\frac{sA k_i^\alpha + (1-\delta)k_i}{1+n}\right].$$ So by observing the values for the parameters $s, \alpha, \delta$ and $n$, and the per-capita capital level $k_i$, we know precisely the level of per-capita capital for the next period, $k_{t+1}$. By using the graph where $k_i$ is on the horizontal and $k_{t+1}$ is on the vertical axis, 45°-line is used, we can graphically investigate the behavior of the economy. Again, we will use $g(k_i)$.

Let’s define $g(k_i)$ as $k_{t+1}$, i.e. $k_{t+1}$, i.e. $g(k_i) = k_{t+1} = \left[\frac{sA k_i^\alpha + (1-\delta)k_i}{1+n}\right]$. 

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If we start at a low capital point at period 0, $k_0$, next period’s capital level $g(k_0) = k_1$ will be higher than today’s capital $k_0$. Therefore, capital increases and goes up to $k_1$ next period. At period 1, now the capital level is $k_1$ and $g(k_1) = k_2$ is still greater than $k_1$, hence capital increases up to $k_2$ next period. As long as $g(k_t) = k_{t+1}$ is still greater than $k_t$ (which are the points where $g(k_t) = k_{t+1}$ lies above the 45°-line) capital keeps increasing. The point where $g(k_t) = k_{t+1} = k_t$, capital remains unchanged, and we have our steady-state.

What happens to the steady-state level of capital as population growth, $n$, increases? $N$ is in the denominator in the $k_{t+1}$ equation, so it will have the same effects as in a decrease in saving rate. That is $g(k_t)$ with higher population growth rate lies below the old $g(k_t)$ at all points, and it intersects the 45°-line at a point to the left of the initial steady-state, implying that steady-state level of per-capita capital and output decreases.

Homework: Show the effect of a higher population growth rate, graphically. In your graph, put $k_t$ is on the horizontal and $k_{t+1}$ is on the vertical axis, and also use the 45°-line.

Hint: Use two different $g(k_t)$ graphs, one with a higher, and one with a lower rate of population growth $n$.

Homework: Find analytical solution for the steady-state level of per-capita capital, steady-state level of per-capita output, steady-state level of per-capita saving, and steady-state level of per-capita consumption for the Solow-model with population growth where the
production function is Cobb-Douglas: $y_t = f(k_t) = Ak_t^\alpha$. To do so, impose steady-state restrictions on the law of motion for per-capita capital $k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n}$.

**Key results:**

The Solow-model predicts that

a) Countries converge to their steady-state levels of per-capita capital and output in the long-run, and if they have different saving rates, depreciation rates, or population growth rates, or productivity levels, the steady-states may differ.

b) Poor countries (countries with lower levels of per-capita capital and output) grow faster.

c) Countries with high saving rates, low depreciation rates, high productivity, and low population growth reach higher levels of steady states.

d) Solow model successfully explains the stylized facts of Kaldor.

e) Still, the model is far from perfect. Countries do keep growing, and the only source of long-run growth can be attributed to growth of productivity $A$, the reason of which is a mystery by itself. The dissatisfaction gave rise to endogenous growth models.